Article

Laplace Transform & the Error Function

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Abstract

We employ the Post-Widder algorithm to determine the inverse Laplace transform of $\frac{1}{s-a\sqrt{s}}$.

Keywords: Error function, Laplace transform, Bateman-Bromwich algorithm, Post-Widder inversion.

1. Introduction

In [1] the authors propose to obtain the following inverse Laplace transform [2]:

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s-a\sqrt{s}}\right], \tag{1}$$

without the application of the Bateman [3, 4]-Bromwich [5] method based in complex variable.

Post [6] and Widder [7, 8] published an interesting expression for the inversion of the Laplace transform $F(s)$:

$$f(t) = lim_{k\to\infty} \frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1}\left(\frac{\partial^k F}{\partial s^k}\right)_{s=\frac{k}{t}}, \qquad t > 0, \tag{2}$$

which is equivalent [9] to the Tuan-Duc procedure [10].

In Sec. 2 we use (2) to solve (1), and we find that $f(t)$ is related with the error function $\text{erf}(x)$ [1, 11].

2. Post-Widder algorithm

If the relation (2) is applied to:

$$F(s) = \frac{1}{s-a\sqrt{s}} = \sum_{r=0}^{\infty} \frac{a^r}{s^{1+\frac{r}{2}}}, \tag{3}$$

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we obtain that:

$$f(t) = lim_{\,k\to\infty}\ \sum_{r=0}^{\infty}\frac{\xi^r}{k!\,k^{\frac{r}{2}}}\prod_{n=1}^{k}\left(\frac{r}{2}+n\right), \qquad \xi = a\sqrt{t}\,, \tag{4}$$

whose even part is given by:

$$f_{even}(t) = lim_{\,k\to\infty}\ \sum_{l=0}^{\infty}\frac{\xi^l}{k!\,k^l}\prod_{n=1}^{k}(l+n) = lim_{k\to\infty}\ \sum_{l=0}^{\infty}\frac{\xi^{2l}}{l!}\prod_{m=1}^{l}\left(1+\frac{m}{k}\right) = \sum_{l=0}^{\infty}\frac{\xi^{2l}}{l!} = e^{\xi^2}, \tag{5}$$

and the corresponding odd part takes the form:

$$f_{odd}(t) = lim_{\,k\to\infty}\ \sum_{q=0}^{\infty}\frac{\xi^{2q+1}}{k!\,k^{q+\frac{1}{2}}}\left(q+\frac{3}{2}\right)_k = \frac{2}{\sqrt{\pi}}\sum_{q=0}^{\infty}\frac{2^{2q}q!}{(2q+1)!}\,\xi^{2q+1}\,, \tag{6}$$

where we employ the relations [11]:

$$\left(q+\frac{3}{2}\right)_k = \frac{\Gamma(q+\frac{3}{2}+k)}{\Gamma(q+\frac{3}{2})}, \qquad \Gamma\left(q+\frac{3}{2}\right) = \frac{\sqrt{\pi}\,(2q+1)!}{2^{2q+1}\,q!}, \tag{7}$$

and for $k \gg 1$:

$$\Gamma\left(q+\frac{3}{2}+k\right) = \frac{\sqrt{2\pi}\,(q+\frac{3}{2}+k)^{q+1+k}}{e^{q+\frac{3}{2}+k}}, \qquad k! = \sqrt{2\pi k}\,\left(\frac{k}{e}\right)^k. \tag{8}$$

Then from (6):

$$f_{odd}(t) = \frac{2}{\sqrt{\pi}}\left(\xi+\frac{2}{3}\xi^3+\frac{4}{15}\xi^5+\frac{8}{105}\xi^7+\ \dots\right), \tag{9}$$

but [11]:

$$\mathrm{erf}(\xi) \equiv \frac{2}{\sqrt{\pi}}\int_0^{\xi} e^{-\eta^2}\,d\eta = \frac{2}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{(-1)^r\xi^{2r+1}}{r!(2r+1)} = \frac{2}{\sqrt{\pi}}\left(\xi-\frac{1}{3}\xi^3+\frac{1}{10}\xi^5-\frac{1}{42}\xi^7+\ \dots\right), \tag{10}$$

whose multiplication by e^{ξ^2} gives (9):

$$f_{odd}(t) = e^{\xi^2}\,\mathrm{erf}(\xi)\,, \qquad \xi = a\,\sqrt{t}\,, \tag{11}$$

hence from (4), (5) and (11):

$$f(t) = e^{\xi^2}[1+\mathrm{erf}(\xi)] = e^{a^2t}\left[1+\mathrm{erf}\left(a\sqrt{t}\right)\right], \tag{12}$$

in harmony with the result of [1].

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