Article

Fractional Field Theory and Physics of the Dark Sector

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Abstract

Derived from the Peccei-Quinn (PQ) mechanism, axions are hypothetical pseudo-Goldstone bosons that restore the charge-parity (CP) symmetry of quantum chromodynamics (QCD). As of today, the mainstream view is that the PQ mechanism offers the most plausible explanation on the puzzle of preserving CP symmetry in QCD. Moreover, several astrophysical models postulate that axions are likely components of Cold Dark Matter (DM). For example, a recent study argues that DM behaves as a strongly coupled superfluid phase consisting of axion-like particles with mass in the eV range or below [1]. Despite these attractive features, experimental searches have either ruled out some axion-based models or placed them under stringent exclusion limits. The object of this work is to show that the concept of spacetime equipped with minimal fractality (the so called *minimal fractal manifold*, MFM in short [2-5]) solves the CP problem of QCD without invoking the PQ paradigm. Rather than discarding axions as superfluous complications of the theory, we conclude that they may be seen as topological signature of the MFM, which we suggestively refer to as "*Cantor Dust*". We tentatively find that the properties of "*Cantor Dust"* match current observations of DM on both cosmological and galactic scales. They also fall in line with the idea that Dark Energy arises from the dynamics of neutrino oscillations on cosmological scales [6].

Keywords: Strong CP problem, Peccei-Quinn mechanism, Axion, Minimal Fractal Manifold, Cold Dark Matter, Bose-Einstein condensate, Cantor dust.

1. Introduction

It is well known that the identity of Dark Matter (DM) remains an outstanding puzzle of contemporary particle physics and cosmology. The existence of DM is amply supported by independent observations, from the rotation trajectories of galaxies, to the dynamics of galaxy clusters, the abundance of light isotopes, gravitational lensing and anisotropic texture of the cosmic microwave background (CMB) radiation. Astrophysical data shows that DM is non-baryonic, collision-less and displays low velocity dispersion.

Recent years have witnessed a substantial surge in theoretical models of DM. Aside from models that interpret DM as outcome of either modified Newtonian gravitation, massive cosmic halo objects (MACHO's) or black holes, there is a vast collection of non-gravitational DM

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candidates, ranging from weakly interacting massive particles (WIMP's), neutralinos, axions, sterile neutrinos, Randall-Sundrum gravitons, strongly interacting massive particles (SIMP's), partially interacting particles, dark photons, color condensates and so on. In particular, the residual light-isotope abundance from primordial nucleosynthesis hints that DM is an exotic particle relic left over from Big Bang. Light axions of mass $10^{-(6-3)}$ eV (dubbed "invisible" axions) were likely frozen out from the early Universe and, as such, are popular candidates for cold DM. For example, light axions may have formed a zero-temperature Bose-Einstein condensate in primordial times which may have persisted as stable phase throughout the evolution of the Universe. Along with other hypothetical light neutral scalar or pseudo-scalar bosons that arise from symmetry breaking (dilatons, familons and Majorons), coupling of axions with normal matter and radiation is suppressed by the scale where the symmetry breaking occurs. As a result, the majority of DM models consider axions to be ultra-weakly interacting and invisible to Standard Model (SM) particles. Over the years, axionic DM has been the subject of extensive theoretical investigations and experimental searches. To zero in on a representative example, a recent study argues that DM behaves as a strongly coupled superfluid phase consisting of axion-like particles with mass in the eV range or below [1]. In yet another example, a heavy axion with mass $M_a \approx 750\,GeV$ is offered as possible interpretation for the di-photon excess reported at the Large Hadron Collider [7].

The object of this work is to show that the concept of spacetime equipped with minimal fractality (the so called *minimal fractal manifold*, MFM in short [2-5]) solves the CP problem of QCD without invoking the PQ paradigm. Rather than discarding axions as superfluous complications of the theory, we conclude that they may be seen as topological signature of the MFM, which we suggestively refer to as "Cantor Dust". We tentatively find that the properties of Cantor Dust fall in line with current observations of DM on both cosmological and galactic scales. They are also compatible with the idea that Dark Energy (DE) arises from the dynamics of neutrino oscillations on cosmological scales [6].

There is growing evidence that the onset of the MFM in close proximity to the electroweak scale has rich and unexplored phenomenological implications [2-5]. To this end, we mention a recent analysis pointing out that multiscale fractality may occur in a range accessible to the LHC scale, yet likely to be directly unobservable at this scale [8].

Our paper is organized as follows: the next section contains a short introduction to the strong CP problem and the PQ mechanism, section 3 develops the solution to the problem using the MFM and the related concept of Cantor Dust. Sections 4 and 5 elaborate on the implications of Cantor Dust in the formation of *topological condensates*, which underlie the physical content of DM and DE. Concluding remarks are presented in the last section. The reader is cautioned that this analysis is entirely preliminary. Due to the evolving nature of research into the physics of the Dark Sector, further work is required to validate, advance or debunk our tentative findings.

2. Brief Overview of the Strong CP Problem

The QCD Lagrangian includes CP violating terms that arise from the non-trivial vacuum structure of the strong interaction. In particular, QCD is affected by instanton effects produced by tunneling among its multiple degenerate vacua. In its basic formulation and omitting color indices for simplicity, the QCD Lagrangian can be cast in the following form

$$L_{QCD} = \overline{\psi}(i\gamma^{\mu}D_{\mu} - me^{i\theta'\gamma_5})\psi - \frac{1}{4}tr(G_{\mu\nu}G^{\mu\nu}) - n_f\frac{g^2\theta}{32\pi^2}tr(G_{\mu\nu}G^{\mu\nu}) \quad (1a)$$

$$G_{\mu\nu} = \partial_{\mu}G_{\nu} - \partial_{\nu}G_{\mu} + gfG_{\mu}G_{\nu} \quad (1b)$$

Here, $\psi, \overline{\psi}$ denote quarks (antiquarks) of mass m, $G_{\mu\nu}$ and $G^{\mu\nu}$ are gluons and their duals, g is the strong interaction coupling, n_f is the number of quark flavors and $f = f^{abc}$, in which $a,b,c = 1,2,...,8$ is the octet of color indices. The first term carries a chiral phase θ' arising from symmetry breaking in the electroweak sector, while the last term contains the QCD vacuum angle θ. After diagonalizing quark masses, the QCD Lagrangian is left with an *effective* CP-violating term

$$L_{\Theta} = \overline{\Theta}(\frac{g^2}{32\pi^2})G_{\mu\nu}G^{\mu\nu} \quad (2)$$

where $-\pi \leq \overline{\Theta} \leq +\pi$. Experimental limits on the neutron dipole moment constrain the CP-violating parameter to exceedingly small values ($\left|\overline{\Theta}\right| \leq 10^{-10}$), even though $\overline{\Theta} = O(1)$ in (2) is otherwise acceptable. This forms the basis for the "*strong CP problem*" which boils down to the question of why the $\overline{\Theta}$ term (2) arises only as a minute contribution to the Lagrangian. To restore the CP invariance of QCD, in 1977 Peccei and Quinn postulated an additional $U(1)_{PQ}$ axial symmetry of the quarks that cancels the "unphysical" term (2). An axion is the pseudo-Goldstone boson of $U(1)_{PQ}$ which breaks this symmetry due to axion coupling to gluons. The resulting Lagrangian in the PQ scenario picks up an extra-term

$$L = (\overline{\Theta} - \frac{\Phi_A}{f_A})\frac{g^2}{32\pi^2}G_{\mu\nu}G^{\mu\nu} \quad (3)$$

in which Φ_A is the axion field and f_A the axion decay constant, the latter being a measure of the energy scale where the $U(1)_{PQ}$ symmetry breaks down. It can be shown that the minimum of the axion potential occurs at $\Phi_A = \overline{\Theta} f_A$, cancels out the $\overline{\Theta}$ term (2) and restores the CP symmetry.

Despite occasional claims to the contrary, there is no irrefutable evidence for axions at the present time. Notwithstanding the null results, the hunt for axions continues in full gear. Additional insights on the PQ mechanism, axion phenomenology and experimental searches can be found in the vast literature dealing with these topics.

3. Dark Matter as Cantor Dust

The starting point of our analysis is the concept of *minimal fractal manifold* (MFM), as introduced and developed in [2-5]. As natural outcome of Renormalization Group theory, MFM is founded on the idea that, near the electroweak (EW) scale $\mu = M_{EW}$, the four-dimensional spacetime turns into a manifold with low fractality, whose local dimension is $D(x) = 4 - \varepsilon(x)$, $\varepsilon(x) << 1$. The onset of $\varepsilon(x)$ leads to a nearly-vanishing deformation of momentum operators acting on quarks, antiquarks and gluons. It is given by

$$\partial^{1-\varepsilon(x)}\psi = [\partial + \varepsilon(x) D_1^1]\psi \tag{4}$$

$$\partial^{1-\varepsilon(x)}\overline{\psi} = [\partial + \varepsilon(x) D_1^1]\overline{\psi} \tag{5}$$

$$\partial^{1-\varepsilon(x)} G_\rho = [\partial + \varepsilon(x) D_1^1] G_\rho,\ \rho = \mu, \nu \tag{6}$$

where

$$D_1^1\psi = \psi^{(1)}(0)\ln(x) + \gamma\,\psi^{(1)}(x) + \int_0^x \psi^{(2)}(x)\ln(x-\sigma)\,d\sigma \tag{7}$$

and their counterpart expressions for antiquarks and gluons. The covariant derivative acting on quarks is accordingly upgraded to

$$D_\mu^{1-\varepsilon(x)} = \partial_\mu^{1-\varepsilon(x)} - i g G_\mu(x) \tag{8}$$

Direct substitution of (4)–(8) into (1) yields the upgraded Lagrangian

$$L_{QCD,\varepsilon(x)} = \overline{\psi}(i\gamma^\mu D_\mu^{1-\varepsilon(x)} - m e^{i\theta'\gamma_5})\psi - \frac{1}{4} tr(G_{\mu\nu,\varepsilon(x)} G^{\mu\nu,\varepsilon(x)}) - n_f \frac{g^2\theta}{32\pi^2} tr(G_{\mu\nu,\varepsilon(x)} G^{\mu\nu,\varepsilon(x)}) \tag{9}$$

in which

$$G_{\mu\nu,\varepsilon(x)} = \partial_{\mu}^{1-\varepsilon(x)} G_{\nu} - \partial_{\nu}^{1-\varepsilon(x)} G_{\mu} + g f G_{\mu} G_{\nu} \tag{10}$$

Developing (9) and (10) and collecting all terms linearly dependent on $\varepsilon(x)$ gives

$$L_{QCD,\varepsilon(x)} = L_{QCD} + \varepsilon(x)\, \mathrm{T}(D_1^1 \psi, D_1^1 G_{\rho}) \tag{11}$$

where $\mathrm{T}(...)$ embodies the overall contribution of low fractality induced by (7). Further assuming that $\varepsilon(x)$ can be reasonably well approximated as a coordinate independent parameter ($\varepsilon(x) \approx \varepsilon$), leads to an extra-term in the QCD Lagrangian having the form

$$\boxed{L_{\varepsilon} = [\varepsilon\, \mathrm{T}(D_1^1 \psi, D_1^1 G_{\rho}) - \overline{\Theta}] \frac{g^2}{32\pi^2} G_{\mu\nu} G^{\mu\nu}} \tag{12}$$

A quick glance at (3) and (12) reveals that the axion and $\varepsilon \mathrm{T}(...)$ play an *identical role* in canceling out the $\overline{\Theta}$ term. Moreover, an attractive feature of (12) is that numerical estimates on ε carry a similar order of magnitude with predictions derived from early proposals for non-integer dimensionality of spacetime. Namely, vacuum fluctuations surrounding an electron in QED have lesser influence in $D < 4$ dimensions. As a result, the first order correction to the anomalous magnetic of the electron in $D = 4 - \varepsilon$ dimensions becomes

$$\delta g = |\varepsilon| \frac{\alpha(\gamma_E + \ln \pi)}{4\pi} \tag{13}$$

where α represents the fine structure constant, γ_E is Euler's constant and the numerical value of parameter ε is found to be

$$|\varepsilon| = (5.3 \pm 2.5) \times 10^{-7} \tag{14}$$

Numerical bounds on ε may be taken from the literature of dimensional regularization models. Using measurements of anomalous magnetic moment of muon and electron, one obtains

$$|\varepsilon| < 10^{-8}, \quad l \propto 10^{-15} m \tag{15}$$

in which l stands for the characteristic length scale for the onset of fractal space-time. Likewise, experimental determination of the Lamb shift in hydrogen yields

$$|\varepsilon| < 10^{-11}, \quad l \propto 10^{-11} m \tag{16}$$

whereas derivations based on astrophysical observations lead to

$$|\varepsilon| < 10^{-9}, \quad l \propto 10^{11} m \quad \text{(planetary precession)} \tag{17}$$

$$|\varepsilon| < 10^{-5}, \; l \propto 14.4\, Gpc \; \text{(cosmic microwave background)} \tag{18}$$

On the same topic, it can be shown that the CMB radiation may be used to set an upper limit for the fractal dimensionality of space [9]. Following this interpretation, we assume below that $|\varepsilon| < \varepsilon_{\max} = O(10^{-5})$.

It is instructive, at this point, to attempt understanding what the physical meaning of (11) and (12) is. Fractal topology may be thought of as a distribution of underlying spacetime fluctuations averaged over a characteristic inverse energy scale (E^{-1}). The logical outcome of this picture is that the second term of (11) "smears out" quantum tunneling among vacua and suppresses instanton propagation, which is typically unrestricted if QCD is embedded in a smooth spacetime background. Likewise, (11) and (12) suggest a novel kind of *topological coupling* between the low fractal texture of spacetime and gluons, which cannot be naïvely reduced to a conventional physical interaction.

The key tenet of [3] is that SM may be pictured as a self-contained *multifractal set* near or above M_{EW}, where each component field of the SM acts as an primary generator of the set. The emergence of this set supplies natural solutions to the *flavor* and *hierarchy puzzles* without invoking unseen properties of the SM below or above the low TeV scale. In light of these intriguing results, it is interesting to investigate how the structure of the set may gradually morph towards a *fully-developed Cantor Dust* in the ultraviolet limit $\varepsilon \to \varepsilon_{\max}$ by raising the running scale significantly above M_{EW} and moving "upstream" along the Renormalization Group flow. We next explore this scenario.

4. Cantor Dust as Superfluid Dark Matter

Proceeding along this path, one finds that is relatively straightforward to apply the mechanism of Bose-Einstein condensation to fully-developed Cantor Dust and link its physical content to the superfluid model of Dark Matter. To this end, we proceed in two steps: we first show how Dark Matter arises as superfluid phase from the Landau-Ginzburg (LG) phenomenology of complex-scalar fields. Then we bridge the gap between *localization* on fractals and the postulated strong-coupling property of superfluid Dark Matter. It is important to keep in mind that, since both steps involve leading-order approximations, they are only relevant as effective field-theoretic constructions.

4.1 Cantor Dust as Bose-Einstein condensate

Let $\varepsilon_0 = \varepsilon_{EW}$ denote the dimensional parameter associated with the electroweak scale $\mu = M_{EW}$. As stated above, the object is to evaluate what happens to the MFM as one runs the RG flow upstream near the ultraviolet limit of the Cantor Dust $\varepsilon = \varepsilon_{\max} << 1$. In doing so, we are looking at a recursive sequence of critical parameters upper limited by $\varepsilon_{\max}$, starting at ε_0 and evolving through running energy scales as $\varepsilon(\mu_{n+1}) > \varepsilon(\mu_n)$, $\mu_{n+1} > \mu_n$, $n = 1, 2, ..., \infty$.

At finite temperature, a superfluid phase may be described as condensate produced by spontaneous breaking of a *U(1)* global symmetry in an underlying field theory [10]. The ensuing analysis proceeds from a handful of assumptions, namely:

A.1) Spacetime dimensionality undergoes inherent fluctuations as $\varepsilon \to \varepsilon_{\max}$.

A.2) Following the conventional prescription of field theory, an "effective" model of fluctuations requires at least one *complex-scalar field* to give meaning to the concepts of "charge" and "current conservation".

A.3) The most natural picture of this field is in the framework of the LG theory, whose applicability extends over the infrared behavior of many dynamic systems.

A.4) As previously alluded to, we take $\varepsilon_{\max} = O(10^{-5})$ based on the maximal deviation of space dimensionality derived in [9].

Building from these premises, we begin with the dimensional complex-scalar fields $\varepsilon(\mu), \varepsilon^{+}(\mu)$ and work out the corresponding LG model in one dimension, where the sliding scale μ is treated as evolution parameter. The analog of the Gross-Pitaevskii classical Lagrangian may be formulated as [10, 12]

$$L_\varepsilon = \partial\varepsilon(\mu)\partial\varepsilon^{+}(\mu) - \lambda_2 \left|\varepsilon(\mu)\right|^2 - \lambda_4 \left|\varepsilon(\mu)\right|^4 \tag{19}$$

with coupling constants $\lambda_{2,4} > 0$. The Lagrangian is invariant under global rotations of the generic form $\varepsilon \to \exp(i\alpha)\varepsilon$, leading to a conserved *U(1)* charge. Since both λ_2 and the quadratic part of the Lagrangian are positive, spontaneous breaking of the global symmetry occurs only if there is a non-vanishing chemical potential associated with the *U(1)* charge [10]. If the magnitude of the chemical potential exceeds λ_2, a Bose-Einstein condensate develops which may be represented as a traditional complex scalar field

$$\varepsilon(\mu) = [\varepsilon_0(\mu) + \varepsilon'_1(\mu) + i\varepsilon'_2(\mu)]\,\frac{\exp[i\zeta(\mu)]}{\sqrt{2}} \tag{20}$$

Here, the chemical potential is absorbed in the definition of the phase $\zeta(\mu)$ and $\varepsilon'_{1,2}$ denote field fluctuations. The equation of motion derived from (19) is given by

$$\partial^2\varepsilon_0(\mu) = \varepsilon_0(\mu)[\sigma^2(\mu) - \lambda_2 - \lambda_4\varepsilon_0^2(\mu)] \tag{21}$$

in which

$$\sigma^2(\mu) = \partial\varepsilon(\mu)\,\partial\varepsilon^+(\mu) \tag{22}$$

and

$$\partial[\varepsilon_0^2(\mu)\partial\zeta(\mu)] = 0 \tag{23}$$

In the zero-temperature limit, the conserved current and stress-energy parameter at the tree-level ($\varepsilon'_{1,2} = 0$) assume the explicit form

$$j_\varepsilon(\mu) = \varepsilon_0(\mu)\partial\zeta(\mu) \tag{24}$$

$$T_\varepsilon(\mu) = [\partial^2\varepsilon_0(\mu) + \varepsilon_0^2(\mu)\partial^2\zeta(\mu) - L_\varepsilon] \tag{25}$$

Relations (24) and (25) may be now mapped to fluid-mechanical current and stress-energy tensor as follows

$$j_\varepsilon(\mu) = n_s \mathrm{v}(\mu) \tag{26}$$

$$T_\varepsilon(\mu) = [T_s(\mu) + p_s(\mu)]\mathrm{v}^2(\mu) - p_s(\mu) \tag{27}$$

in which n_s, T_s and p_s stand for the charge density, energy density and pressure measured in the superfluid rest-frame. Their respective covariant expressions may be generically presented as

$$n_s = \mathrm{v}(\mu) j_\varepsilon(\mu)\,, \quad T_s = \mathrm{v}^2(\mu) T_\varepsilon\,, \quad p_s = -\frac{1}{3}[1 - \mathrm{v}^2(\mu)]\,T_\varepsilon \tag{28}$$

(26) and (27) enable one to match the superfluid pictures of Cantor Dust and of the axionic Dark Matter developed in [1].

4.2 Wave localization on Cantor Dust

In the last two decades, wave transport in disordered or fractal media has been the focus of extensive research in both condensed matter and engineering applications. In what follows, we briefly latch onto this knowledge and look at the mechanism of localization on fully-developed Cantor Dust. The goal is to evaluate under what circumstances wave localization resembles the behavior of strongly-coupled field theory. A good starting point is the propagation of a generic perturbation on a linear fractal support using scalar wave equation [11]

$$\rho_n(\varepsilon_{n+1}-\varepsilon_n)-\rho_{n-1}(\varepsilon_n-\varepsilon_{n-1})+\omega^2\varepsilon_n=0 \tag{29}$$

The key ingredient of this approach is the nearly continuous "distribution" of mass scales within the fully-developed Cantor Dust, which it is assumed to take the form

$$\rho_n=\rho_0(1+\eta_n) \tag{30}$$

$$\rho_n=\sum_i\frac{(m_n)_i}{M_{0,n}},\quad \rho_0=\sum_i\frac{(m_0)_i}{M_{EW}} \tag{31}$$

Here, n is index of the running scale, $(m_n)_i$ denote the sequence of masses forming the multifractal set at the reference scale $M_{0,n}$ and η_n stands for the fluctuation in the mass distribution. It is seen that, by this definition, $M_{0,0}=M_{EW}$. The tacit assumption here is that the reference scale $M_{0,n}$ runs along with the dimensional parameter $\varepsilon_n=4-D_n$.

We next introduce the plausible assumption that fully-developed Cantor Dust corresponds to the emergence of a Weierstrass type distribution of scales near the UV limit, as $\varepsilon\to\varepsilon_{\max}$ and $\mu_{\max}>>M_{EW}$. The fluctuation η_n is then expected to take the form [11]

$$\eta_n=\sum_k^{\infty}\cos[\gamma^k\frac{n}{L_n}+\varphi_k]/\gamma^{(2-\Delta)k} \tag{32}$$

in which Δ stands for the box-counting dimension and $L_n=L(\mu_n)$ for the (normalized) linear size of the system

$$L_n=\frac{M_{0,n}}{M_{EW}} \tag{33}$$

The box-counting dimension Δ controls the magnitude of localization effects on the fully-developed Cantor Dust. On the basis of (32), analysis shows that wave transport on fully-developed Cantor Dust is strongly inhibited for $\Delta > 3/2$ [11]. If this condition is satisfied, localization of wave transport on the dust emulates the confining attributes of strongly coupled field theory [see, e.g. 13].

5. Cantor Dust and Dark Energy

It was shown in [2-5] that neutrinos arise as nearly-massless states towards the lower end of the MFM as spacetime dimensionality asymptotically nears four ($\varepsilon \to 0, D = 4$). This picture is consistent with the interpretation of the DE and cosmological constant in terms of neutrino oscillations on cosmological scales [6]. Following this line of reasoning, DE may be seen as manifestation of Cantor Dust in the limit of vanishing fractality, where MFM makes transition to the conventional spacetime continuum. *An appealing consequence of this view is that both DM and DE are hidden aspects of the same topological structure, the Cantor Dust.*

6. Conclusions

A benchmark model of the Dark Sector based on the concept of Cantor Dust has been outlined. Our tentative findings may be summarized as follows:

1) spacetime equipped with nearly-vanishing fractality ($\varepsilon << 1$) solves the CP problem of QCD without invoking the PQ paradigm,

2) fully-developed Cantor Dust emerging near $\varepsilon_{\max} = O(-10^{-5})$ replicates the physical attributes of the axionic superfluid model of DM,

3) wave localization on fully-developed Cantor Dust resembles the confining behavior of strongly-coupled field theory,

4) Cantor Dust is also compatible with the idea that DE arises from the dynamics of neutrino oscillations on cosmological scales, thus providing a unified picture of the Dark Sector.

Abbreviations
PQ = Peccei-Quinn
QCD = Quantum Chromodynamics
CP = charge-parity symmetry
SM = the Standard Model of high-energy physics
MFM = Minimal Fractal Manifold
DM = Dark Matter
DE = Dark Energy

CD = Cantor Dust
EW = Electroweak
LG = Landau-Ginzburg

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