Forum

Energy Is Conserved in the Classical Theory of General Relativity

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Abstract

The first law of thermodynamics states that energy is conserved. It is one of the most fundamental laws of physics and not one that you would expect many physicists to challenge, so it comes as a surprise to find that a growing number of cosmologists and relativists are doing just that. Of course any law of physics is subject to experimental verification and as new realms of observation are opened up we should require that previous assumptions including conservation of energy are checked. But the subject under question is not new physics in this sense. It is the classical theory of general relativity. Whether general relativity is correct is not the issue, although it has withstood all experimental tests so far. The question concerns whether energy is conserved in the classical theory of general relativity with or without cosmological constant as given be Einstein nearly 100 years ago. This is a purely mathematical question.

Key Words: general relativity, energy-momentum, conservation, classical theory, Einstein.

I. Energy Is Conserved In GR

It has indeed been said that too much ink has been spilt on this subject already, but the fact is that the wrong conclusions are still drawn. It does not matter how well-respected the cosmologists are or how many people have read their textbooks, the fact is that they are wrong. Energy *is* conserved in general relativity. There are no ifs or buts. The mathematics is clear and the errors in the thinking of those who think it is not conserved can also be traced. It is time to put the record straight.

Not all the cosmologists are so bold as to state directly that energy is not conserved, but some are. Here are some examples of the kind of things they do say:

"*there is not a general global energy conservation law in general relativity theory*" – Phillip Peebles in Principles of Physical Cosmology

"*In special cases, yes. In general — it depends on what you mean by 'energy', and what you mean by 'conserved'*." – John Baez and Michael Weiss in the Physics FAQ

"*The energy conservation law is an identity in general relativity*" - Felix Klein

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"*the local conservation laws, integrated over a closed space [..] produce nothing of interest, only the trivial identity 0 = 0*", John Wheeler in Geometrodynamics.

"*a local energy density is well-defined in GR only for spacetimes that admit a timelike Killing vector*" – Steve Carlip on sci.physics.research

"*Energy is Not Conserved*" – Sean Carroll on Cosmic Variance

"*Energy is not conserved in cosmology. As is always the case with confusing stuff in cosmology, this is covered well in Edward Harrison's COSMOLOGY textbook*." – Phillip Helbig on usenet

Many other statements have been made to the effect that conservation of energy in general relativity is only approximate, quasi-local, trivial, non-covariant, ambiguous or only valid in special cases. They are all wrong. Energy is conserved in general relativity.

Discussions about conservation of energy in cosmology often arise when people write about redshift of the cosmic background radiation. Individual photons are not created or destroyed as they travel across space, so if they are redshifted they are losing energy. Where does it go? The answer is that it goes into the background gravitational field. The presence of the CMB affects slightly the rate at which the universe is expanding so there should be an energy term for the expansion rate. This is what happens for particles moving in other types of background such as electric and magnetic fields so it should work for gravity too.

Since the discovery of "dark energy" the level of confusion has become worse. People visualise dark energy as a constant density of energy that pervades space. If space expands then there should be more of it, so where does the energy come from? The anser is the same as for radiation. The dark energy, or cosmological constant as it used to be known, affects the expansion rate of the universe. The gravitational component of energy has a contribution from this exapnsion and the rate changes to conteract the amount of dark energy being added so that total energy is constant.

Noether's Theorem

To make the case for energy conservation in general relativity sound, we need a valid mathematical formula for it in terms of the gravitational field (the metric tensor) and the matter fields. This problem was initially tackled as soon as general relativity was proposed by Einstein. The mathematician Emmy Noether was asked to look at the problem and she solved it eloquently by stating her theorem relating symmetry to conservation laws. Although the theorem is well-known to physicists it is not often appreciated that it was formulated to tackle this specific problem.

Noether's theorem tells us that if a physical law derived from an action principle is invariant under time translations, then it has an energy conservation law. In fact the theorem provides a formula to derive an energy current whose divergence is zero. Such a current can always be integrated over a region space to provide a total energy whose rate of change is equal to the flux of the current from the surface bounding the region. This is exactly what we mean by conservation of energy.

For example if we take Maxwell's equations in special relativity such invariance applies and we can derive a formula for the energy current. Of course in special relativity time is not absolute and there are different concepts of time dependent on an observers velocity. This means that we actually get an infinite number of energy conservation laws, one for each possible velocity. Conveniently this boils down to a single energy-momentum tensor that gives the energy current for any choice of the time coordinate in a reference frame. The same tensor can be used to provide momentum and angular momentum conservation laws. It is all very intuitive and nice!

Energy-momentum pseudotensors

What about the case of general relativity? Invariance under time translation still holds and general relativity is derived from the Hilbert action principle so Noether's theorem can be applied to the gravitational field along with any matter fields to give a total conserved energy current, but there is a technical hitch. The Hilbert action includes second derivatives of the metric tensor as well as the first, and Noether's theorem only deals with the case where there are first derivatives. The usual solution applied in the early days of relativity was to modify the Hilbert action in a way that removed the terms containing the second derivatives without affecting the dynamics of the Einstein equations derived from it. Noether's theorem could then be applied. The only snag was that the procedure could not be made gauge invariant so the energy-momentum quantities derived did not form a covariant tensor as they did for special relativity. Sometimes they are called the energy-momentum pseudotensor. The solution works but some people just don't like it. They complain that the pseudo-tensor can be made zero at any point in spacetime for example. It is not really a problem but people did not expect it so they complain about it.

The source of the problem (which is not really a problem) can be traced to the fact that the spacetime symmetry group in general relativity is bigger than it is in special relativity. Instead of just a choice of time coordinate for each velocity of an inertial reference frame, you have one for any choice of motion whether inertial or not. This gives a much larger set of conservation laws and with the extra choice you can always make the energy and momentum of the field zero at any given event in space and time.

The choice of time coordinate can be associated with a contravariant vector field that generates the time translation. We should expect the formula for our energy from Noether's theorem to have a dependency on this field. Trying to express it as a tensor is not really appropriate and that is what causes the confusion.

Modern Covariant Solution

It turns out that there is a more general version of Noether's theorem that can be used even when the action includes terms with second derivatives. This provides a more modern approach to the derivation of an energy current that has a dependency on the time translation vector field. Since it does not require any manipulations of the action the result is a covariant local expression. I am avoiding formulae here but you can look up, the answer in arXiv:gr-qc/9701028 which is now published in this journal (1). This paper does not take into account the cosmological constant but that is not a problem. The conditions for Noether's theorem still apply with the cosmological constant term in place and the derivation of this more general case is a straightforward exercise left for the reader.

So the outcome is that there is a local covariant expression for the energy current in general relativity after all. This is exactly the thing that many cosmologists claim does not exist, but it does, and energy conservation holds perfectly with no caveats.

To finish off let's take a look at some of the specific things that cosmologists and relativists have been saying and debunk them one by one in the light of the solution we now understand.

Energy Conservation in general relativity is approximate NOT

It is sometimes claimed that energy conservation in general relativity is only approximate. On further examination of what is meant we find that the person who thinks this only knows of (or only accepts as valid) the extension of the covariant energy-momentum tensor from special relativity to the general theory. This tensor includes only contributions from the matter fields and not the gravitational field. Its covariant divergence is zero just as required for a conserved current vector, but unfortunately it is a symmetric tensor and you can not integrate a divergenceless symmetric tensor to get a conserved quantity in curved spacetime. That only works for vectors and anti-symmetric tensors. Because of this people say that the conservation is only approximate.

It should be clear now where the error in this argument lies. The energy-momentum tensor does not include contributions from the gravitational field and energy conservation cannot be formulated without it. Of course your energy conservation law is only going to be approximate if you neglect one of the fields that has energy.

The correction is to include the gravitational field either by using the pseudotensor method or by using the more modern derivation of the current as a function of the time translation vector field.

Energy conservation only works in special cases in general relativity NOT

The cause of this false claim is once again the use of the energy-momentum tensor. For some special cases the gravitational field has a killing vector that indicates that it is static in some specific reference frame. If you contract this killing vector with the energy-momentum tensor you get an expression for an energy current that is conserved. That's very nice but nothing unusual. It is normal that you can get a conserved energy in a fixed background field which is static. The same happens for other fields such as the electromagnetic field. If the energy in the background field in not changing then the energy in the rest of the system can be conserved too without adding the energy from the background field.

Just because energy conservation is a bit simpler in special cases does not mean that it does not work in more general cases, which it does of course.

Another special case often cited is an asymptotically flat spacetime. You can work out the total energy and momentum and it takes the form of a familiar energy-momentum four vector in the asymptotic limit. Very nice, but again just a special case while the general case also works perfectly well.

Energy conservation in general relativity is trivial NOT

This particular version of the energy conservation "problem" in general relativity goes back to the early days when Noether, Einstein, Klein, Hilbert and others were investigating it. Klein claimed that the conservation law that Noether's theorem gave was an identity, so there was no real physical content to the law. This claim has been echoed many times since, for example when Wheeler claimed that the law reduces to the trivial result 0 = 0 for closed spacetimes.

In addition to her well-known theorem, Noether had a second theorem that elaborated on what happens when there is a local gauge symmetry rather than just a global symmetry. In this case you can derive Bianchi type identities that provide formulae for currents that are conserved kinematically, even if the equations of motion are not. You can say that such a current is trivially conserved. The formula for the energy current derived from Noether's theorem is not such a quantity, but it is the sum of two parts one of which is trivially conserved and the other of which is always zero when the field equations apply. For some people this is enough to make the claim that energy conservation is trivial in general relativity.

That this makes no sense is easily seen by considering any other gauge field and its conserved charges. For example, electromagnetism is a gauge theory that conserves electric charge. Because of Noethers second theorem the expression for the electric current can be written as the sum of a term depending only on the electromagnetic potential whose divergence is explicitly zero, plus a term which is obviously zero when Maxwell's equations hold. This is exactly analogous to the case of energy conservation in general relativity. Nobody claims that this makes charge conservation trivial in the classical theory of electromagnetism so they should not make such a claim for energy conservation in general relativity.

Conclusion

I have debunked some of the major claims about energy conservation in general relativity that people use to justify the idea that there is something wrong with it. There are others but they are all just as shallow and easy to deal with. If you come across anyone making such claims, please just refer them to here and hopefully we can put an end to this nonsense.

II. Energy Is Conserved in GR (the Math)

Judging by the comments on the previous article I have not yet succeeded in convincing anyone that energy is conserved. Luboš Motl has posted a response contradicting my viewpoint and agreeing with an older blog post by Sean Carroll. Fortunately I have an advantage, I've done the maths and the outcome is clear and unambiguous. Since my detractors are people who understand equations I should have no trouble convincing them if I take a more technical approach. So, no more analogies, let's start with Einstein's equation.

$$G_{ab} + \Lambda g_{ab} = \kappa T_{ab}$$

G_{ab} is the Einstein tensor given by

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$$

R_{ab} is the Ricci curvature tensor, g_{ab} is the metric tensor, Λ is the cosmological constant, κ is a gravitational coupling constant and T_{ab} is the energy-stress tensor for matter.

A time translation is generated by a time-like vector field ξ^a with a small parameter ϵ

$$\Delta x^a = \epsilon \xi^a$$

All the tensor quantities have corresponding transformation rules and the field equations are covariant under the transformation. Using Noether's theorem a formula for the corresponding conserved energy current can be derived. The details of this calculation are quite lengthy and can be found in arXiv:gr-qc/9701028 (without the cosmological constant which is easy to add in. see Ref. 1) The result obtained is

$$J^a = J^a_M + J^a_{DE} + J^a_G$$

Where J^a_M is the energy current from the matter contribution given by

$$J^a_M = \xi^b T_{cb} g^{ca}$$

J^a_{DE} is the dark energy component given by

$$J^a_{DE} = \frac{-1}{\kappa} \xi^a \Lambda$$

and J^a_G is the gravitational contribution given by

$$J^a_G = \frac{-1}{\kappa} \xi^b \, G_{cb} g^{ca} + K^a$$

K^a is the Komar superpotential given by

$$K^a = \frac{1}{2\kappa} (\xi^b{}_;{}^a - \xi^a{}_;{}^b)_{;b}$$

Given the Einstein field equations we can eliminate the matter term, the dark energy term and the first part of the gravitational term to leave just the Komar superpotential

$$J^a = K^a$$

Then it is easy to see that

$$J^a{}_{;a} = K^a{}_{;a} = 0$$

Since the current is divergenceless it defines a conserved energy. Some people claim that this result is trivial. Clearly it is not because it requires the gravitational field equations to prove conservation of energy. You should not make the mistake of defining energy as just the Komar superpotential, even though it is equal to that when the dynamics are taken into account. Energy must be defined as a sum over contributions from each field including the gravitational field, and dark energy.

The energy contribution from matter $J^a_M = \xi^b T_{cb} g^{ca}$ is a sum of contributions from each type of matter field. It includes all the non-gravitational forms of energy including heat, electrical energy, rest mass equivalent of energy, radiation etc. This form of the law of energy conservation in general relativity tells us that the energy from all these contributions plus the gravitational contribution and the dark energy contribution is conserved. In other words energy can be transformed from one form to another but is never created or destroyed. There is nothing approximate, trivial or ambiguous about this result. It is energy conservation in the same old form that we have always known it but with the contribution from gravity included.

III. Energy Is Conserved in GR (the History)

We have been discussion the law of conservation of energy in the context of classical general relativity. So far I have not been able to convince anyone here that the maths shows that energy is conserved. Lubos Motl and Matti Pitkanen have posted some contrary arguments on their blogs to add to the old one by Sean Carroll. We have also been trading points and counterpoint in the comments with Ervin Goldfain joining in, also in disagreement with me. To avoid going over the same arguments repeatedly we have agreed to disagree, for now.

If you think such a discussion about Energy in physics seems off the wall, think again. This subject and related issues concerning gravitational waves have occupied physicists for years. Some well-known names in the world of science have exchanged some heated words and still not everyone agrees on the outcome.

But it is too soon to end our debate. There are still a few more points I want to make. It was said that my claim in favour of energy conservation means that I am "convinced that all relativists are wrong". This is not the case. Historically many relativists have been on my side. This is actually a debate that began as soon as general relativity was formulated by Einstein. Einstein in fact developed the first complete formulation of energy conservation in GR, but Hilbert objected. The argument has raged ever since with as many different views on the subject as there have been relativists and cosmologists. Amongst those who have accepted the law of energy conservation and produced their own formulations are Dirac, Landau, Wald, Weinberg and of course Einstein himself, so to say I am contradicting all relativists is far from true.

It has also been said that all the textbooks show that energy is not conserved in general relativity, except in special cases. This is also not true. Most GR textbooks do not tackle the general formulation of energy conservation in GR. They just deal with special cases such as a static background gravitational field with a killing vector. This does not mean that energy conservation only works in special cases as some people claim. The textbooks just don't cover the general case. Some textbooks do cover it but by using pseudotensor methods (e.g. Dirac, Weinberg, Landau & Lifshitz) A few textbooks do suggest that energy is not conserved, e.g. Peebles, but these are the minority.

I am going to recount some of the history of the debate. To keep it orderly I'll give it as a timeline of events with my own contribution immodestly tacked on the end. We start in 1915 with conservation of energy a well established concept recently unified with the conservation of mass by Einstein. The World is at war and Einstein is about to publish his general theory.

July 1915: Einstein lectures on his incomplete theory of general relativity to Hilbert, Klein and possible Noether at Göttingen, convincing them that his ideas are important.

October 1915: Albert Einstein publishes a tentative equation for general relativity $R_{ab} = T_{ab}$ with R_{ab}being the Ricci curvature tensor and T_{ab}being the covariant generalization of the energy-momentum tensor.

November 1915: Einstein realises that his previous equation cannot be right because the divergence of the energy momentum tensor is zero as required by local energy conservation. To correct it he writes the new equation $R_{ab} - \frac{1}{2}g_{ab} = T_{ab}$. These are the Einstein Field equations which work because the left hand side has zero divergence due to the Bianchi identities.

November 1915: David Hilbert publishes a calculation showing how the Einstein Field Equations can be derived from a least action principle. In fact his work is dated prior to Einstein's but they had been in communication and it is reasonable to give the priority for the equations to Einstein and for the action formulation to Hilbert.

1916: Hilbert publishes a note with an equation for a conserved energy vector in general relativity.

1916: Einstein publishes a full formulation of energy conservation in general relativity in which a pseudotensor quantity is added to the energy-momentum tensor and another superpotential term to give a conserved energy current.

1916: Einstein predicts the existence of gravitational waves which will carry away energy and momentum from orbiting stars. He derives the quadrupole radiation formula to quantify the rate at which energy is dispersed.

July 1917: Oscar Klein points out (with help from Noether) that conservation of Hilbert's energy vector is an identity that does not require the field equation.

1917: In response to Klein, Hilbert publishes an article questioning the validity of energy conservation in general relativity. He says that the energy equations do not exist at all and this is a general characteristic of the theory.

1917: Writing to Klein, Hilbert says that general relativity has only improper energy theorems. By this he means that the pseudotensor methods are not covariant.

1917: Klein writes to Einstein making the claim that energy conservation in general relativity is an identity. This is based on Hilbert's energy vector.

1917: To construct a static cosmological model Einstein introduced the cosmological constant as an extra term in his field equations.

March 1918: Einstein writes back to Klein explaining that in his formulation of energy conservation the divergence of the current is not an identity because it requires the field equations.

July 1918: Emmy Noether publishes two theorems on symmetry in physics. The first showed that symmetry in any theory derived from an action principle implies a conservation law. In particular, energy conservation is implied by time invariance. The second shows that in the case of gauge theories with local symmetry such as general relativity, there are divergence identities such as the Bianchi Identities.

1918: Felix Klein uses Noether's theorems to derive a third boundary theorem to show why the conservation law of energy in general relativity must take a particular form that he considers to make it an identity.

1918: Einstein comments on the power and generality of Noether's theorems but does not accept the conclusion that energy conservation is an identity.

1919: Arthur Eddington measures the deflection of starlight by the Sun during a solar eclipse. The observation confirms the prediction of general relativity and provides massive press publicity for the theory.

1922: Arthur Eddington expresses skepticism about the existence of gravitational waves saying that they "travel at the speed of thought".

1922: Friedman finds cosmological solutions of general relativity that describe an expanding universe.

1927: Lemaitre predicts an expanding universe

1929: Edwin Hubble observes the expanding universe in galactic redshifts. This led to Einstein dropping his cosmological constant.

1936: After working on exact solutions for gravitational waves with Rosen, Einstein concludes that gravitational waves can not exist, reversing his 1916 prediction. This sparked a vigorous twenty year debate over the reality of gravitational waves.

1936: After working with Robertson, Einstein eventually concedes that gravitational waves do exist. Rosen who had departed for the Soviet Union did not accept this concession. He never changed his mind even as late as 1970.

1951: Landau and Lifshitz publish "The Classical Theory of Fields" as part of a series of textbooks on theoretical physics. It deals with energy and momentum in general relativity using a symmetric pseudotensor. The symmetry means that they can also show conservation of angular momentum using the same structure.

1955: Rosen computes the energy in exact gravitational wave solutions using pseudotensors and finds the result ot be zero. He presents this as evidence that gravitational waves are not real.

1957: Herman Bondi introduced a formalism now known as Bondi Energy to study energy in general relativity and gravitational waves in particular. This work was very influential and formed a turning point in the understanding of gravitational waves and energy in general realtivity.

1957: Weber and Wheeler find a gravitational wave solution that does transmit energy.

1957: Richard Feynman describes the sticky-bead thought experiment to show that gravitational waves are real. The idea was popularised by Herman Bondi and finally led to the general acceptance of the reality of gravitational waves.

1959: Andrzej Trautman gave the formulation of energy conservation for the special case where a static background is given by the existence of a killing vector field.

1959: Komar defined a superpotential for general cases whose divergence vanishes as an identity. The superpotential uses an auxiliary vector field similar to the killing vector field in Trautmans theory, but the Komar field does not need to satisfy any special conditions so the solution is more general. The Komar potential has the advantage over pseudotensor methods that it is expressed in a covariant form. However, the zero divergence of the superpotential is an identity.

1961: Arnowitt, Deser and Misner formulate the ADM mass/energy for systems in asymptotically flat spacetimes

1961: In his book "Geometrodynamics" Archibald Wheeler says that energy conservation in a closed universe reduces to a trivial 0 = 0 equation.

1964: Weber begins experiments to try to detect gravitational waves.

1972: Steven Weinberg uses a pseudotensor method to show energy conservation in his textbook "Gravitation and Cosmology"

1974: The discovery of the Hulse-Taylor binary pulsar shows that gravitational energy is radiated as originally predicted by Einstein.

1975: In his concise introductory text to general relativity Dirac derives a pseudotensor using Noether's theorem to prove energy and momentum conservation.

1979: Schoen and Yau prove the positive energy theorem for ADM energy. (A simpler proof was given by Witten in 1981)

1993: Phillip Peebles in his book "Principles of Physical Cosmology" claims that energy conservation is violated for the cosmic microwave background

1997: Philip Gibbs shows that if Noether's theorem is generalised to include second derivatives of the fields and is applied to the symmetries generated by a vector field, a conserved current with a covariant current can be derived. The current which has an explicit dependence on the vector field is equal to a term that is zero when the Einstein Field Equations are satisfied, plus the Komar superpotential.

1998: Observational evidence (Riess, Pulmutter) leads to the reintroduction of the cosmological constant, now called dark energy

For further historical details and references on the Klein-Hilbert-Einstein-Noether debate see "A note on General Relativity, Energy Conservation and Noether's Theorems" by Katherine Brading in "The Universe of General Relativity" ed A.J. Fox, J. Eisenstaedt.

A good read on the history of gravitational waves is "Traveling at the speed of thought" by Daniel Kennefick

III. Energy Is Conserved in GR (in Cosmology)

On viXra log we have been having some lengthy discussions on energy conservation in classical general relativity. I have been trying to convince people that Energy *is* conserved, but most of them who have expressed an opinion think that energy is not conserved, or that the law of conservation of energy is somehow trivial in general relativity with no useful physical content.

I am going to have one more try to show why energy is conserved and is not trivial by tackling the question of energy conservation in cosmology. Some physicists have claimed that energy conservation is violated when you look at the cosmic background radiation. This radiation consists of photons that are redshifted as the universe expands. The total number of photons remains constant but their individual energy decreases because it is proportional to their frequency ($E = hf$) and the frequency decreases due to redshift. This implies that the total energy in the radiation field decreases, but if energy is conserved, where does it go? The answer is that it goes into the gravitational field, but to make this answer convincing we need some equations.

If the radiation question is not strong enough, what about the case of the cosmological constant, also known as dark energy? With modern precision cosmological observation it is now known that the cosmological constant is not zero and that dark energy contributes about 70% of the total non-gravitational energy content of the observable universe at the current cosmological epoch. (We assume here a standard cosmological model in which the dark energy is a fixed constant and not a dynamic field.) As the universe expands, the density of dark energy stays constant. This means that in an expanding region of space the total dark energy must be increasing. If energy is conserved, where is this energy coming from? Again the answer is that it comes from the gravitational field, but we need to look at the equations.

These are questions that surfaced relatively recently. As I mentioned in my history post, the original dispute over energy conservation in general relativity began between Klein, Hilbert and Einstein in about 1916. It was finally settled by about 1957 after the work of Landau, Lifshitz, Bondi, Wheeler and others who sided with Einstein. After that it was mostly discussed only among science historians and philosophers. However, the discovery of cosmic microwave background and then dark energy have brought the discussion back, with some physicists once again doubting that the law of energy conservation can be correct.

Energy in the real universe has contributions from all physical fields and radiation including gravity and dark energy. It is constantly changing from one form to another, it also flows from one place to another. It can travel in the form of radiation such as light or gravitational waves. Even the energy loss of binary pulsars in the form of gravitational waves has been observed indirectly and it agrees with experiment. None of these processes is trivial and energy is conserved in all cases. But what about energy on a truely universal scale, how does that work?

On scales larger than the biggest galactic clusters, the universe has been observed to be very close to homogeneous and isotropic. Furthermore, 3 dimensional space is flat on average as far as we can tell, and it is expanding uniformly. Spacetime curvature and gravitational energy on these large scales comes purely from the expansion component of space as a function of time. The metric for this universe is

$$ds^2 = a(t)^2 ds_3^2 - c^2 dt^2$$

$$ds_3^2 = dx^2 + dy^2 + dz^2$$

$a(t)$ is an expansion factor that increases with time (For full details see http://en.wikipedia.org/wiki/Friedmann_equations)

I gave previously the equation for the Noether current in terms of the fields and an auxiliary vector field that specifies the time translation diffeomorphisms. The Noether current has a term called the Komar superpotential but for the standard cosmology this is zero. The remaining terms in the zero component of the current density come from the matter fields and the spacetime curvature and are given by

$$J^0 = \rho + \frac{\gamma}{a} + \frac{\Lambda c^2}{\kappa} - 3\frac{\dot{a}^2}{\kappa a^2}$$

The first term is the mass-energy from cold matter, (including dark matter) at density ρ. The second term is the energy density from radiation. The third term is dark matter energy density and the last term is the energy in the gravitational field. Notice that the gravitational energy is negative. By the field equations we know that the value of the energy will be zero. This equation is in fact one of the Freidmann equations that is used in standard cosmology.

If you prefer to think of total energy in an expanding region of spacetime rather than energy density, you should multiply each term of the equation by a volume factor a^3

It should now be clear how energy manages to be conserved in cosmology on large scales even with a cosmological constant. The dark energy in an expanding region increases with the volume of the region that contains it, but at the same time the expansion of space accelerates exponentially so that the negative contribution from the gravitational field also increases in magnitude rapidly. The total value of energy in an expanding region remains zero, and therefore constant. This is not a trivial result because it is equivalent to the Friedmann equation that captures the dynamics of the expanding universe.

So there you have it; the cosmological energy conservation equation that everybody has been asking about is just this

$$E = Mc^2 + \frac{\Gamma}{a} + \frac{\Lambda c^2}{\kappa}a^3 - \frac{3}{\kappa}\dot{a}^2 a = 0$$

It is not very complicated or mysterious, and it's not trivial because it describes gravtational dynamics on the scale of the observable universe.

In this equation

- $a(t)$is the universal expansion factor as a funcrtion of time normalised to 1 at the current epoch.
- $M$is the total mass in the expanding volume $V = a(t)^3$
- $\Gamma$is the cosmic radiation energy density fixed at the current epoch
- $\Lambda$is the cosmological constant.
- $\kappa$is a gravitational coupling constant.

Reference

1. Philip E. Gibbs, *Covariant Energy-Momentum Conservation in General Relativity with Cosmological Constant*, Prespacetime Journal V1(6): pp 899-907. Also see http://arxiv.org/abs/gr-qc/9701028 .