Article

Einstein Spaces of Class One

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Abstract

The Einstein spaces R_3 and R_4 are umbilical when they are locally and isometrically embedded (class one) into a Riemannian space of constant curvature is proved.

Keywords: Embedding of 4-spaces of Riemann; Gauss-Codazzi equations.

1. Introduction

Here we deal with Riemannian spaces of three and four dimensions of Einstein type, that is, those where the Ricci tensor R_{ac} is proportional to the metric tensor g_{ac}:

$$R_{jk} = \frac{R}{n} g_{jk}, \quad n = 3, 4 \tag{1}$$

If these spaces are locally and isometrically embedded into another (n+1)-dimensional Riemannian space of constant curvature K, then the Gauss-Codazzi equations are verified [1-6]:

$$R_{ijrm} - K(g_{ir}g_{jm} - g_{im}g_{jr}) = \varepsilon\,(b_{ir}b_{jm} - b_{im}b_{jr}), \tag{2}$$

$$b_{ij;r} - b_{ir;j} = 0, \tag{3}$$

where $\varepsilon \pm 1$, R_{acij} is the Riemann tensor of R_n, $b_{ic} = b_{ci}$ is the corresponding second fundamental form and ; r means the covariant derivative. In Sec. 2 we shall use expressions obtained in [7-11] to show that (1, 2) imply the umbilical character of R_n, that is:

$$b_{jc} = \frac{b}{n} g_{jc}, \qquad b \equiv b^r_{\ r}, \qquad n = 3,4. \tag{4}$$

In fact, (4) is correct for every n, which may be seen [12] by a careful analysis of the eigenvalue problem for b_{jr}. By substitution of (4) into (2) one can deduce the relation:

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$$R_{ijrm} = \overline{K}\,(g_{ir}g_{jm} - g_{im}\,g_{jr}), \qquad \overline{K} = K + \frac{\varepsilon b^2}{n^2} \tag{5}$$

So, the treated R_n, $n = 3,4$ also turns to be a space of constant gaussian curvature $\overline{K}$.

2. Umbilical property of R_n, n = 3, 4 of Einstein space embedded into (n+ 1) - space of constant curvature

Equation (2) was studied in [8] only for the case $K = 0$, however, we can extend the use of the same scheme to our main analysis without major difficulty; when $K \neq 0$ we obtain the relation:

$$pb_{ij} = \frac{R}{6} + \frac{1}{6}R_i^m R_{mj} - \frac{1}{3}R_{imrj}R^{mr} - \frac{1}{12}R_{imrc}R_j^{mrc} + \frac{n(n-1)}{6}KR_{ij} + \\ + \frac{K}{6}\left[(n-3)R + n(n-1)(n-2)K\right]g_{ij}, \tag{6}$$

with

$$p = \frac{\varepsilon}{3}b^{rc}G_{rc} - \frac{\varepsilon K}{6}(n-1)(n-2)b, \tag{7}$$

where $R = R^j{}_j$ is the scalar curvature and $G_{ij} = R_{ij} - \frac{R}{2}g_{ij}$ is the Einstein tensor of R_n.

It is now convenient to split our analysis in two directions:

a). Case n = 3

Here we shall see the condition that implies the umbilical character of R_3. It is widely known [1, 2] that in three dimensions the Ricci tensor generates the Riemann tensor:

$$R_{ijrc} = R_{jr}g_{ic} + R_{ic}g_{jr} - R_{ir}g_{jc} - R_{jc}g_{ir} + \frac{R}{2}(g_{ir}g_{jc} - g_{ic}g_{jr}) \tag{8}$$

hence the introduction of (1, 8) into (6, 7) gives (4, 5) with:

$$b^2 = -9\varepsilon\left(K + \frac{R}{6}\right) > 0, \qquad \overline{K} = -\frac{R}{6} \qquad \text{if} \quad \left(K + \frac{R}{6}\right) \neq 0 \tag{9}$$

which determines the sign of ε.

b). Case n = 4

Lanczos identities [13] reduce (6, 7) to the form [7-11, 14, 15]:

$$pb_{ij} = -\frac{1}{2} R_{ircj} G^{rc} + 2KR_{ij} + (4K^2 + \frac{KR}{6} + \frac{1}{48} K_2) g_{ij} \tag{10}$$

such that

$$p = \frac{\varepsilon}{3} b^{ic} G_{ic} - \varepsilon b K, \qquad K_2 = {}^* R^{*ijrc} R_{ijrc} \tag{11}$$

where $*R*arjc$ is the double dual [2, 4, 5, 7, 13] of the Riemann tensor. By substitution of (1) into (10, 11) we obtain (4, 5) with:

$$b^2 = -16\varepsilon (K + \frac{R}{12}) > 0, \qquad \overline{\mathrm{K}} = -\frac{\mathrm{R}}{12}, \qquad K_2 = -\frac{R^2}{6}, \tag{12}$$

under the condition $(K + \frac{R}{12}) \neq 0$; this result (12) can be seen as a generalization of the theorem II of [16].

In this way we have showed that our relations (6, 7) give a simple proof of the umbilical character of R_3 (R_4) of Einstein embedded into R_4 (R_5) of constant curvature.

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