Article

On Two Results for the Terminating $_3F_2$ (2)

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Abstract

We exhibit elementary proofs of the formulae for $_3F_2(-N, a, 1+\lambda; 1+2a, \lambda; 2)$ obtained by Kim-Choi-Rathie.

Keywords: Hypergeometric functions, Kim-Choi-Rathie's equations.

1. Introduction

Kummer [1] proved the identity:

$$_1F_1(\beta; 2\beta; x) = e^{\frac{x}{2}}\, {}_0F_1\left(\beta + \frac{1}{2}; \frac{x^2}{16}\right), \tag{1}$$

which implies the values [2, 3]:

$$_2F_1(-M, \beta; 2\beta; 2) = \begin{cases} \frac{(\frac{1}{2})_n}{(\beta+\frac{1}{2})_n}, & M = 2n, \quad (2) \\ 0 & M = 2n-1, \quad (3) \end{cases}$$

in terms of the Pochhammer [4]-Barnes [5, 6] symbol (shifted factorial [7]).

Kim-Choi-Rathie [8] obtained the formulae:

$$_3F_2(-N, a, 1+\lambda; 1+2a, \lambda; 2) = \begin{cases} \frac{(\frac{1}{2})_n}{(a+\frac{1}{2})_n}, & N = 2n, \quad (4) \\ \frac{\left(1-\frac{2a}{\lambda}\right)(\frac{3}{2})_n}{(1+2a)(a+\frac{3}{2})_n}, & N = 2n+1, \quad (5) \end{cases}$$

In the next Section we employ (2) and (3) to give elementary proofs of (4) and (5).

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2. Kim-Choi-Rathie's expressions for $_3F_2(2)$

With the definition of the hypergeometric function $_3F_2$ [9] is simple to deduce the interesting relation:

$$\frac{d}{d\lambda}\ {}_3F_2(-N,a,1+\lambda;1+2a,\lambda;2)=\frac{2aN}{(1+2a)\lambda^2}\ {}_2F_1(1-N,1+a;2+2a;2), \tag{6}$$

hence it is convenient to consider two cases:

$\boldsymbol{N=2n}$ Then from (6):

$$\frac{d}{d\lambda}\ {}_3F_2(-2n,a,1+\lambda;1+2a,\lambda;2)=\frac{4an}{(1+2a)\lambda^2}\ {}_2F_1(-(2n-1),1+a;2(1+a);2)=0,$$

by (3) with $M=2n-1$ and $\beta=1+a$. Therefore $_3F_2(-2n,a,1+\lambda;1+2a,\lambda;2)$ is independent of the parameter λ, and thus we can calculate it via some value of λ, for example $\lambda=2a$:

$$ {}_3F_2(-2n,a,1+\lambda;1+2a,\lambda;2)=\ {}_3F_2(-2n,a,1+2a;1+2a,2a;2)=\ {}_2F_1(-2n,a;2a;2),$$

$$=\frac{(\frac{1}{2})_n}{(a+\frac{1}{2})_n}=(4),\ \ q.e.d.$$

where we use the result (2) with $M=2n$ and $\beta=a$.

$\boldsymbol{N=2n+1}$ From (6):

$$\frac{d}{d\lambda}\ {}_3F_2(-2n-1,a,1+\lambda;1+2a,\lambda;2)=\frac{2a(2n+1)}{(1+2a)\lambda^2}\ {}_2F_1(-2n,1+a;2(1+a);2),$$

$$\overset{(2)}{=}\frac{2a(2n+1)(\frac{1}{2})_n}{(1+2a)(a+\frac{3}{2})_n\lambda^2}=\frac{2aQ}{\lambda^2}\ ,\qquad Q=\frac{(\frac{3}{2})_n}{(1+2a)(a+\frac{3}{2})_n}\ ,$$

whose integration gives:

$$ {}_3F_2(-2n-1,a,1+\lambda;1+2a,\lambda;2)=-\frac{2aQ}{\lambda}+A\ , \tag{7}$$

where A is independent of λ. Then we can determine A with $\lambda=2a$:

$$ {}_3F_2(-2n-1, a, 1+2a; 1+2a, 2a; 2 = {}_2F_1(-2n-1, a, 2a; 2) \overset{(3)}{=} 0 \overset{(7)}{=} -Q + A, $$

that is, $A = Q$, hence (7) implies (5), $q.e.d.$

Our process shows that the Kim-Choi-Rathie's expressions can be deduced, in elementary manner, from the relations (2), (3) and (6).

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