Article

Faraday Tensor & Maxwell Spinor (Part I)

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Abstract

We realize an algebraic study of the spinors associated with real and Newman-Penrose tetrads in Minkowski space and obtain an adequate platform for the spinorial analysis of Faraday and energy-momentum tensors of the electromagnetic field. Maxwell spinor naturally appears in our approach.

Part I of this two-part article includes: 1. Introduction, 2. The Cartan spinor, and 3. Tetrads and their 2-spinors.

Key words: null tetrad, Faraday tensor, 2-spinors, Maxwell equations, Pauli matrices, quaternions, Maxwell spinor, Lorentz transformations.

1. Introduction

In Minkowski spacetime, we have the metric tensor $(g_{\mu\nu}) = Diag\,(1,-1,-1,-1)$ with the quaternion of position:

$$\mathbf{R} = \frac{1}{\sqrt{2}}(c\,t + i\,x\,\mathbf{I} + i\,y\,\mathbf{J} + i\,z\,\mathbf{K}), \qquad i = \sqrt{-1}\,, \tag{1}$$

and the corresponding Lorentz transformations are generated by means of the Klein-Sommerfeld's expression [1-6]:

$$\widetilde{\mathbf{R}} = \mathbf{A}\,\mathbf{R}\,\overline{\mathbf{A}}^*\,, \tag{2}$$

where $\mathbf{A} = a_0 + a_1\,\mathbf{I} + a_2\,\mathbf{J} + a_3\,\mathbf{K}$ is an arbitrary unit quaternion, and:

$$\overline{\mathbf{A}}^* = a_0^* - a_1^*\,\mathbf{I} - a_2^*\,\mathbf{J} - a_3^*\,\mathbf{K}\,. \tag{3}$$

The formula (2) was obtained by Hamilton [7] and Cayley [8] for 3-rotations [9, 10], in such a case the quantities $a_\mu,\ \mu = 0, \dots, 3$ are real and do match with the Euler-Olinde Rodrigues parameters [11-13].

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The matrix version of (2) will be the starting point of our spinorial analysis, and it can be deduced by means of the isomorphism introduced by Cayley [14, 15] between the quaternion basis elements and the Cayley [14]-Sylvester [16]-Pauli [17] matrices:

$$1 \leftrightarrow \mathrm{I}_{2x2}\,, \qquad \mathbf{I} \leftrightarrow -i\,\sigma_1\,, \qquad \mathbf{J} \leftrightarrow -i\,\sigma_2\,, \qquad \mathbf{K} \leftrightarrow -i\,\sigma_3\,, \tag{4}$$

so that

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

Thus, the quaternion **A** is isomorphic to the complex matrix 2x2 [11, 18]:

$$\mathrm{A} = \begin{pmatrix} a_0 - ia_3 & -a_2 - ia_1 \\ a_2 - ia_1 & a_0 + ia_3 \end{pmatrix} = a_0\,\mathrm{I} - i\,a_1\sigma_1 - i\,a_2\sigma_2 - i\,a_3\sigma_3\,, \tag{6}$$

moreover

$$\mathbf{A}\,\overline{\mathbf{A}} = a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1 \qquad \Longrightarrow \qquad det\ \mathrm{A} = 1\,, \tag{7}$$

as is the case when it is employed (2) to build Lorentz transformations. With the aid of (6), it is straightforward to find the matrices associated with the quaternions (1) and (3):

$$\mathbf{R} \leftrightarrow \mathrm{X} = \frac{1}{\sqrt{2}}\begin{pmatrix} ct+z & x-iy \\ x+iy & ct-z \end{pmatrix}, \qquad \overline{\mathbf{A}}^* \leftrightarrow \mathrm{A}^\dagger = \mathrm{A}^{\mathrm{T}^*} = \begin{pmatrix} a_0^* + ia_3^* & a_2^* + ia_1^* \\ -a_2^* + ia_1^* & a_0^* - ia_3^* \end{pmatrix}, \tag{8}$$

then (2) leads to the Cartan's expression [19, 20]:

$$\widetilde{\mathrm{X}} = \mathrm{A}\,\mathrm{X}\,\mathrm{A}^\dagger\,, \qquad det\ \mathrm{A} = det\,\mathrm{A}^\dagger = 1\,, \tag{9}$$

also obtained by Olinde Rodrigues [21] for 3-rotations.

Taking the determinant of (9) we deduce the conservation of Minkowski's interval:

$$c^2\tilde{t}^2 - \tilde{x}^2 - \tilde{y}^2 - \tilde{z}^2 = c^2t^2 - x^2 - y^2 - z^2\,, \tag{10}$$

which implies [12] the linearity of the coordinate transformation between both reference frames (we use the summation convention on repeated indices introduced by Dedekind (1868) [23, 24] and Einstein):

$$\tilde{x}^\mu = \mathrm{L}^\mu{}_\nu\, x^\nu, \qquad (x^\mu) = (ct, x, y, z), \tag{11}$$

where $\mathrm{L} = (\mathrm{L}^\alpha{}_\beta)$ is an element of the homogeneous Lorentz group with $det\,\mathrm{L} = 1$. The substitution of (11) into (9) provides explicit formulas for $\mathrm{L}^\mu{}_\nu$ in terms of the Euler-Olinde Rodrigues parameters, see [4, 22, 25, 26]. The matrices ±A lead to the same L, therefore they constitute a bi-representation of the Lorentz transformations.

In (9) we have the 2-spinor (Ehrenfest introduced the term spinor, see [27-29]):

$$\left(\mathrm{X}^{\mathrm{A}\dot{\mathrm{B}}}\right) = \begin{pmatrix} \mathrm{X}^{1\dot{1}} & \mathrm{X}^{1\dot{2}} \\ \mathrm{X}^{2\dot{1}} & \mathrm{X}^{2\dot{2}} \end{pmatrix}, \qquad \mathrm{X}^{1\dot{1}} = \frac{1}{\sqrt{2}}(x^0 + x^3), \qquad \mathrm{X}^{1\dot{2}} = \frac{1}{\sqrt{2}}(x^1 - i\,x^2), \tag{12}$$

$$\mathrm{X}^{2\dot{1}} = \frac{1}{\sqrt{2}}(x^1 + i\,x^2), \qquad \mathrm{X}^{2\dot{2}} = \frac{1}{\sqrt{2}}(x^0 - x^3), \qquad \overline{\mathrm{X}^{\mathrm{A}\dot{\mathrm{B}}}} = \mathrm{X}^{\mathrm{B}\dot{\mathrm{A}}},$$

it is evident the Hermitian character of X; furthermore, one of its indices transforms (under a Lorentz mapping) according to A, meanwhile the dotted index does so via $\mathrm{A}^{\dagger}$:

$$\widetilde{\mathrm{X}^{\mathrm{B}\dot{\mathrm{C}}}} = \mathrm{A}^{\mathrm{B}}{}_{\mathrm{D}}\,\mathrm{X}^{\mathrm{D}\dot{\mathrm{E}}}\,\mathrm{A}^{\dagger}{}_{\dot{\mathrm{E}}}{}^{\dot{\mathrm{C}}}, \tag{13}$$

which, together with (12), is equivalent to the tensorial relation (11) due to the connection of L with $\mathrm{A} = (\mathrm{A}^{\mathrm{B}}{}_{\mathrm{C}})$ and $\mathrm{A}^{\dagger} = \left(\mathrm{A}^{\dagger}{}_{\dot{\mathrm{B}}}{}^{\dot{\mathrm{C}}}\right)$.

The paper is organized as follows. In Sec. 2 it is undertaken a detailed study of the spinor X, which allows to introduce in a natural manner the Infeld-van der Waerden symbols $\sigma_{\mu}{}^{\mathrm{A}\dot{\mathrm{B}}}$ [30, 31], of great importance to perform the spinor transcription of a tensor expression, or analogously, given a certain tensor, to deduce its corresponding spinor. We show that these symbols provide an explicit formula for L in terms of A and $\mathrm{A}^{\dagger}$, in harmony with the results obtained in [32]. On the other hand, it is established that for x^{μ} null it is possible to express $\mathrm{X}^{\mathrm{A}\dot{\mathrm{B}}}$ as the product of simple spinors, which turns out to be relevant in the spinorial analysis of vectors on the light cone.

In Sec. 3, we consider the simple spinors associated with an arbitrary null tetrad of Newman-Penrose (NP) type [33], which in turn generates spinors for a real orthonormal Minkowskian tetrad, that facilitates the spinorial study of any tensor (for instance, the skew-symmetric tensor of the Maxwell field) written in terms of a real tetrad, or in terms of a NP type. In [34-37], it was used a real tetrad to build a basis for any skew-symmetric tensor of second order, with the aim of analyzing the trajectories of charged particles (with or without radiation reaction) in special relativity.

Finally, in Sec. 4 we apply this technique to the electromagnetic tensor, and it is shown the existence of the Maxwell symmetric spinor. The method and the results of this section are successfully applied to obtain the spinorial structure of Maxwell's energy-momentum tensor. In addition, in Secs. 2-4 we also indicate the NP versions of the main spinorial relations.

2. The Cartan spinor

In (12) it is immediate the following expansion:

$$\mathrm{X} = \frac{x^0}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{x^1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \frac{x^2}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + \frac{x^3}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{14}$$

which motivates the introduction of the Infeld-van der Waerden symbols [30, 31, 38, 39] in terms of the Cayley-Sylvester-Pauli matrices indicated in (5):

$$\left(\sigma_0{}^{A\dot{B}}\right) = \frac{1}{\sqrt{2}}\, \mathrm{I}\,, \qquad \left(\sigma_j{}^{A\dot{B}}\right) = \frac{1}{\sqrt{2}}\, \sigma_j\,, \qquad j = 1, 2, 3, \tag{15}$$

notice that it is verified the Hermitian property $\overline{\sigma_\mu{}^{A\dot{B}}} = \sigma_\mu{}^{B\dot{A}}$. Then (14) acquires the form:

$$X^{A\dot{B}} = x^\mu\, \sigma_\mu{}^{A\dot{B}}\,, \qquad x^\mu \;\leftrightarrow\; X^{A\dot{B}}\,, \tag{16}$$

whose structure shows the pattern to follow for constructing the 2-spinor associated with a vector, for each tensor index we have a pair of spinor indices. In $\sigma_\mu{}^{A\dot{B}}$ the μ index can be raised with the Minkowski metric $\left(g_{\mu\nu}\right) = Diag\,(1, -1, -1, -1)$, thus:

$$\left(\sigma^{0\,A\dot{B}}\right) = \frac{1}{\sqrt{2}}\, \mathrm{I}\,, \qquad \left(\sigma^{k\,A\dot{B}}\right) = -\frac{1}{\sqrt{2}}\, \sigma_k\,, \qquad k = 1, 2, 3, \tag{17}$$

then it is easy to prove that:

$$\sigma^{\mu\,A\dot{B}}\, \sigma_\mu{}^{C\dot{D}} = \epsilon^{AC}\, \epsilon^{\dot{B}\dot{D}}\,, \tag{18}$$

with the skew-symmetric matrices:

$$\left(\epsilon^{AB}\right) = \left(\epsilon_{AB}\right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \left(\epsilon^{\dot{A}\dot{B}}\right) = \left(\epsilon_{\dot{A}\dot{B}}\right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\epsilon_{AB}\, \epsilon^{CD} = \delta_A^C\, \delta_B^D - \delta_A^D \delta_B^C\,, \tag{19}$$

$$\epsilon_B{}^D = -\epsilon^D{}_B = \epsilon_{AB}\, \epsilon^{AD} = \delta_B^D\,, \qquad \epsilon_{AB}\, \epsilon^{AB} = 2\,;$$

in [40] it is indicated that ϵ_{AC} is the quantity that defines the sympletic complex structure of the spin space.

The spinorial indices can be raised and lowered by means of (19) (we shall employ the Penrose-Rindler convention [41]):

$$\psi^A = \epsilon^{AB}\, \psi_B\,, \qquad \psi_C = \epsilon_{BC}\, \psi^B\,, \qquad \psi^{\dot{A}} = \epsilon^{\dot{A}\dot{B}}\, \psi_{\dot{B}}\,, \qquad \psi_{\dot{C}} = \epsilon_{\dot{B}\dot{C}}\, \psi^{\dot{B}}\,, \tag{20}$$

that is, for an arbitrary simple spinor:

$$\psi^1 = \psi_2\,, \quad \psi^2 = -\psi_1 \quad \therefore \quad \psi^A \psi_A = 0\,, \quad \psi^{\dot{A}} \psi_{\dot{A}} = 0\,, \quad \psi^A \phi_A = -\psi_A \phi^A\,. \tag{21}$$

Furthermore,

$$\left(\sigma^0{}_{A\dot{B}}\right) = \left(\sigma_{0\,A\dot{B}}\right) = \frac{1}{\sqrt{2}}\, \mathrm{I}\,, \qquad \left(\sigma^1{}_{A\dot{B}}\right) = \left(-\sigma_{1\,A\dot{B}}\right) = -\frac{1}{\sqrt{2}} \sigma_1\,, \tag{22}$$

$$\left(\sigma^2{}_{A\dot{B}}\right) = \left(-\sigma_{2\,A\dot{B}}\right) = -\frac{1}{\sqrt{2}} \sigma_2\,, \qquad \left(\sigma^3{}_{A\dot{B}}\right) = \left(-\sigma_{3\,A\dot{B}}\right) = -\frac{1}{\sqrt{2}} \sigma_3\,,$$

and together with (15) it implies the interesting relation [42]:

$$\sigma_\mu{}^{A\dot{B}}\,\sigma_{\nu\,A\dot{B}} = g_{\mu\nu}\,, \tag{23}$$

that allows to invert the equation (16):

$$x^\mu = \sigma^\mu{}_{A\dot{B}}\,X^{A\dot{B}}\,, \tag{24}$$

where it is shown how to obtain the vector associated with a 2-spinor, indeed, the Infeld-van der Waerden symbols capture one pair of spinor indices to assign one tensor index.

The numerical values (15) are not altered with a change of reference frame:

$$\sigma^{\mu\,B\dot{C}} = L^\mu{}_\nu\,A^B{}_D\,\sigma^{\nu\,D\dot{E}}\,A^\dagger{}_{\dot{E}}{}^{\dot{C}}\,, \tag{25}$$

it is not difficult to invert the matrices (6) and (8), see [41]:

$$(A^{-1\;B}{}_C) = \begin{pmatrix} a_0 + ia_3 & a_2 + ia_1 \\ -a_2 + ia_1 & a_0 - ia_3 \end{pmatrix}, \qquad \left(A^{\dagger\,-1}{}_{\dot{B}}{}^{\dot{C}}\right) = \begin{pmatrix} a_0^* - ia_3^* & -a_2^* - ia_1^* \\ a_2^* - ia_1^* & a_0^* + ia_3^* \end{pmatrix}, \tag{26}$$

then, using (23) and (26) in (25) the Lorentz matrix is deduced in terms of the Olinde Rodrigues parameters and of the Infeld-van der Waerden symbols:

$$L^\mu{}_\nu = \sigma^\mu{}_{D\dot{E}}\;\sigma_\nu{}^{B\dot{C}}\;A^{-1\,D}{}_B\;A^{\dagger\,-1}{}_{\dot{C}}{}^{\dot{E}}\,, \tag{27}$$

from which the expressions of [32] are immediate and the explicit formulas of [22, 25, 26] for L in terms of the aforesaid parameters.

With the prescription (16) and the identity (18) we can build the spinor associated with the metric tensor:

$$g_{AC\dot{B}\dot{D}} = \sigma^\mu{}_{A\dot{B}}\,g_{\mu\nu}\,\sigma^\nu{}_{C\dot{D}} = \sigma^\mu{}_{A\dot{B}}\,\sigma_{\mu\,C\dot{D}} = \epsilon_{AC}\,\epsilon_{\dot{B}\dot{D}}\,, \tag{28}$$

compatible with (23), and from (24) we obtain $x^\mu x_\mu = x^\mu x^\nu g_{\mu\nu} = X^{A\dot{B}}X^{C\dot{D}}\epsilon_{AC}\epsilon_{\dot{B}\dot{D}}$, therefore:

$$x^\mu x_\mu = X^{A\dot{B}}X_{A\dot{B}}\,. \tag{29}$$

If the 2-spinor $X^{D\dot{E}}$ is the product of two simple spinors:

$$X^{A\dot{B}} = \xi^A\,\xi^{\dot{B}}, \quad X_{2x2} = (\xi^A)_{2x1}\left(\xi^{\dot{B}}\right)_{1x2},$$

$$(\xi^A) = \begin{pmatrix}\xi^1\\ \xi^2\end{pmatrix}, \;\; \left(\xi^{\dot{B}}\right) = \left(\xi^{\dot{1}}\;\;\xi^{\dot{2}}\right) = \left(\overline{\xi^1}\;\;\overline{\xi^2}\right) = \begin{pmatrix}\xi^1\\ \xi^2\end{pmatrix}^\dagger, \tag{30}$$

then, its associated vector must be null because with (21) and (30) we have that $X^{A\dot{B}}\,X_{A\dot{B}} = \left|\xi^A\xi_A\right|^2 = 0$, thus (29) gives:

$$c^2t^2 - x^2 - y^2 - z^2 = x^\mu x_\mu = 0\,, \tag{31}$$

x^ν is on a light cone, and it can be assumed that it is pointing to the future $(x^0 > 0)$. Now, the aim is to calculate ξ^{A} given $\mathrm{X}^{\mathrm{A}\dot{\mathrm{B}}}$:

$$\xi^1 = p\, e^{i\varphi}\,, \qquad \xi^{\dot{1}} = p\, e^{-i\varphi}\,, \qquad \xi^2 = q\, e^{i\theta}\,, \qquad \xi^{\dot{2}} = q\, e^{-i\theta}\,, \tag{32}$$

and the expressions:

$$p^2 = \xi^1\,\xi^{\dot{1}} = \mathrm{X}^{1\dot{1}} = \tfrac{1}{\sqrt{2}}(x^0 + x^3), \qquad q^2 = \xi^2\,\xi^{\dot{2}} = \mathrm{X}^{2\dot{2}} = \tfrac{1}{\sqrt{2}}(x^0 - x^3), \tag{33}$$

are univocally determining the magnitudes p and q, however, θ and φ are arbitrary except for their difference which can be obtained by means of:

$$pq\, e^{i(\varphi-\theta)} = \tfrac{1}{\sqrt{2}}(x^1 - ix^2), \tag{34}$$

this manifests the non-unicity of ξ^{A} due to the fact that it can be multiplied by an arbitrary phase without altering the Cartan 2-spinor:

$$\mathrm{X}^{\mathrm{A}\dot{\mathrm{B}}} = \xi^{\mathrm{A}}\,\xi^{\dot{\mathrm{B}}} = \xi^{\mathrm{A}}\,\overline{\xi^{\mathrm{B}}} = \left(e^{i\Omega}\,\xi^{\mathrm{A}}\right)\left(\overline{e^{i\Omega}\,\xi^{\mathrm{B}}}\right). \tag{35}$$

If x^μ points to the past $(x^0 < 0)$, the decomposition will take the form $\mathrm{X}^{\mathrm{A}\dot{\mathrm{B}}} = -\xi^{\mathrm{A}}\,\xi^{\dot{\mathrm{B}}}$. In summary, if k^ν is a real null vector, then:

$$k^\mu \qquad \leftrightarrow \qquad \mathrm{K}^{\mathrm{A}\dot{\mathrm{B}}} = \gamma^{\mathrm{A}}\,\gamma^{\dot{\mathrm{B}}}\,, \tag{36}$$

where γ^{C} is defined up to an arbitrary phase.

With (36) and $t^\mu \leftrightarrow \mathrm{T}^{\mathrm{A}\dot{\mathrm{B}}} = \eta^{\mathrm{A}}\eta^{\dot{\mathrm{B}}}$ one gets the inner product:

$$k^\mu\, t_\mu = \left|\gamma_{\mathrm{A}}\eta^{\mathrm{A}}\right|^2, \tag{37}$$

therefore $k^\nu t_\nu = 0$ if and only if $\gamma_{\mathrm{A}}\eta^{\mathrm{A}} = 0$, but Synge [22] demonstrates that $k^\nu t_\nu = 0$ implies the proportionality of such null vectors, thus:

$$k^\mu = \lambda\, t^\mu \qquad \Leftrightarrow \qquad \gamma_{\mathrm{A}}\eta^{\mathrm{A}} = 0, \tag{38}$$

in whose case $\gamma_{\mathrm{A}} = \sqrt{\lambda}\,\eta_{\mathrm{A}}$. When studying the Newman-Penrose tetrad and the electromagnetic field, a situation with $k^\mu t_\mu = 1$ arises, and the arbitrarity in the phases of γ^{A} and η^{B} allows to choose the norm $\gamma_{\mathrm{A}}\eta^{\mathrm{A}} = -\gamma^{\mathrm{A}}\eta_{\mathrm{A}} = 1$, in harmony with (37).

In Sec. 3, the analysis made here (for the Cartan $X^{A\dot{B}}$) is extended to a null tetrad of the NP type [33] and to its corresponding real orthonormal tetrad, which is important in the spinorial structure of the Faraday and Maxwell tensors (Sec. 4).

3. Tetrads and their 2-spinors

For each event in the spacetime it can be constructed a real orthonormal tetrad:

$$e_{(0)}{}^{\mu}\, e_{(0)\,\mu} = 1, \qquad e_{(0)}{}^{\mu}\, e_{(j)\,\mu} = 0, \qquad e_{(j)}{}^{\nu}\, e_{(k)\,\nu} = -\delta_{jk}, \qquad j,k = 1,2,3 \tag{39}$$

positive-oriented:

$$\eta_{\mu\nu\alpha\beta}\, e_{(0)}{}^{\mu}\, e_{(1)}{}^{\nu}\, e_{(2)}{}^{\alpha}\, e_{(3)}{}^{\beta} = 1, \tag{40}$$

where the totally skew-symmetric Levi-Civita tensor takes part:

$$\eta^{\mu\nu\alpha\beta} = -\eta_{\mu\nu\alpha\beta} = 1 \text{ or } -1 \text{ if } (\mu\nu\alpha\beta) \tag{41}$$

is even or odd permutation of (0123), respectively, and 0 if two of its indices have the same value.

The real tetrad permits to establish a basis for any tensorial object, for example, the electromagnetic field tensor (Sec. 4), thus the spinorial study of (39) is useful in the deduction of the Maxwell spinor and does also provide a convenient platform for the spinor formulation of differential geometry of curves [43, 44]; besides, it leads to the Newman-Penrose null tetrad [33, 45-47]:

$$l^{\mu} = \frac{1}{\sqrt{2}}\left(e_{(0)}{}^{\mu} + e_{(3)}{}^{\mu}\right), n^{\mu} = \frac{1}{\sqrt{2}}\left(e_{(0)}{}^{\mu} - e_{(3)}{}^{\mu}\right),$$

$$m^{\mu} = \frac{1}{\sqrt{2}}\left(e_{(1)}{}^{\mu} - i e_{(2)}{}^{\mu}\right), \bar{m}^{\mu} = \frac{1}{\sqrt{2}}\left(e_{(1)}{}^{\mu} + i e_{(2)}{}^{\mu}\right), \tag{42}$$

with the properties:

$$l^{\mu} n_{\mu} = -m^{\nu}\bar{m}_{\nu} = 1, \quad l^{\mu} m_{\mu} = n^{\nu} m_{\nu} = 0,$$

$$l^{\mu} l_{\mu} = n^{\mu} n_{\mu} = m^{\mu} m_{\mu} = 0, \quad \eta_{\mu\nu\alpha\beta} l^{\mu} n^{\nu} m^{\alpha} \bar{m}^{\beta} = -i. \tag{43}$$

According to (36), the real null vectors n^{μ} and l^{μ} have got associated simple spinors, which we shall denote with the Greek letters iota and omicron, respectively:

$$l^{A\dot{B}} = o^{A} o^{\dot{B}}, \qquad n^{A\dot{B}} = \iota^{A}\iota^{\dot{B}}, \qquad o_{A}\iota^{A} = -o^{A}\iota_{A} = 1\,. \tag{44}$$

The vector m^{μ} is associated with the spinor $m^{A\dot{B}}$ that can be written in terms of a basis of simple spinors:

$$m^{C\dot{D}} = \lambda_1\, o^{C} o^{\dot{D}} + \lambda_2\, \iota^{C}\iota^{\dot{D}} + \lambda_3\, o^{C}\iota^{\dot{D}} + \lambda_4\, \iota^{C} o^{\dot{D}}\,,$$

but (43) imposes conditions, for example, $o_{\mathrm{C}} o_{\dot{\mathrm{D}}} m^{\mathrm{C}\dot{\mathrm{D}}} = 0$ implies $\lambda_2 = 0$, $\iota_{\mathrm{C}} \iota_{\dot{\mathrm{D}}} m^{\mathrm{C}\dot{\mathrm{D}}} = 0$ gives $\lambda_1 = 0$, $m_{\mathrm{C}\dot{\mathrm{D}}} m^{\mathrm{C}\dot{\mathrm{D}}} = 0$ leads to $\lambda_3 \lambda_4 = 0$, and $m_{\mathrm{C}\dot{\mathrm{D}}} \overline{m^{\mathrm{D}\dot{\mathrm{C}}}} = -1$ requires that $|\lambda_3|^2 + |\lambda_4|^2 = 1$, then without loss of generality we choose $\lambda_3 = 1$ and $\lambda_4 = 0$, therefore:

$$m^{\nu} \ \leftrightarrow \ m^{\mathrm{C}\dot{\mathrm{D}}} = o^{\mathrm{C}} \iota^{\dot{\mathrm{D}}} , \qquad \overline{m}^{\nu} \ \leftrightarrow \ \mathrm{M}^{\mathrm{C}\dot{\mathrm{D}}} = \overline{m^{\mathrm{D}\dot{\mathrm{C}}}} = \iota^{\mathrm{C}} o^{\dot{\mathrm{D}}} , \tag{45}$$

which together with (42) and (44) gives the spinors associated with the real tetrad:

$$\mathrm{e}_{(0)\,\nu} \ \leftrightarrow \ \frac{1}{\sqrt{2}}(o_{\mathrm{A}} o_{\dot{\mathrm{B}}} + \iota_{\mathrm{A}} \iota_{\dot{\mathrm{B}}}), \qquad \mathrm{e}_{(1)\,\nu} \ \leftrightarrow \ \frac{1}{\sqrt{2}}(o_{\mathrm{A}} \iota_{\dot{\mathrm{B}}} + \iota_{\mathrm{A}} o_{\dot{\mathrm{B}}}), \tag{46}$$
$$\mathrm{e}_{(2)\,\nu} \ \leftrightarrow \ \frac{i}{\sqrt{2}}(o_{\mathrm{A}} \iota_{\dot{\mathrm{B}}} - \iota_{\mathrm{A}} o_{\dot{\mathrm{B}}}), \qquad \mathrm{e}_{(3)\,\nu} \ \leftrightarrow \ \frac{1}{\sqrt{2}}(o_{\mathrm{A}} o_{\dot{\mathrm{B}}} - \iota_{\mathrm{A}} \iota_{\dot{\mathrm{B}}}),$$

consistent with (39), and taking into account the norm indicated in (44).

With the real and NP tetrads, it is straightforward to generate the metric tensor:

$$g_{\mu\nu} = l_{\mu} n_{\nu} + l_{\nu} n_{\mu} - m_{\mu} \overline{m}_{\nu} - m_{\nu} \overline{m}_{\mu} = \mathrm{e}_{(0)\,\mu} \mathrm{e}_{(0)\,\nu} - \mathrm{e}_{(j)\,\mu} \mathrm{e}_{(j)\,\nu} , \qquad j = 1, 2, 3 \tag{47}$$

where we can use (44) and (45) or (46) to deduce the spinor associated with the Minkowski metric:

$$g_{\mathrm{AC}\dot{\mathrm{B}}\dot{\mathrm{D}}} = (o_{\mathrm{A}} \ \mathrm{x} \ \iota_{\mathrm{C}})\,(o_{\dot{\mathrm{B}}} \ \mathrm{x} \ \iota_{\dot{\mathrm{D}}}), \tag{48}$$

with the Lowry notation [48] applicable to tensorial and spinorial indices:

$$\mathrm{A}_{\mu} \ \mathrm{x} \ \mathrm{B}_{\nu} \ \equiv \ \mathrm{A}_{\mu}\,\mathrm{B}_{\nu} - \mathrm{A}_{\nu}\,\mathrm{B}_{\mu} , \tag{49}$$

and comparing this with (28) we get the expressions:

$$\epsilon_{\mathrm{AB}} = o_{\mathrm{A}} \ \mathrm{x} \ \iota_{\mathrm{B}} = o_{\mathrm{A}} \iota_{\mathrm{B}} - o_{\mathrm{B}} \iota_{\mathrm{A}} , \qquad \epsilon_{\dot{\mathrm{B}}\dot{\mathrm{D}}} = o_{\dot{\mathrm{B}}} \ \mathrm{x} \ \iota_{\dot{\mathrm{D}}} , \tag{50}$$

which are valid for any pair of simple spinors that fulfill the normalization (44); in particular, by means of (50) it is immediate to obtain the useful relation:

$$2(o_{\mathrm{A}} \iota_{\mathrm{C}} o_{\dot{\mathrm{B}}} \iota_{\dot{\mathrm{D}}} - o_{\mathrm{C}} \iota_{\mathrm{A}} o_{\dot{\mathrm{D}}} \iota_{\dot{\mathrm{B}}}) = \epsilon_{\mathrm{AC}}\,(o_{\dot{\mathrm{B}}} \iota_{\dot{\mathrm{D}}} + o_{\dot{\mathrm{D}}} \iota_{\dot{\mathrm{B}}}) + (o_{\mathrm{A}} \iota_{\mathrm{C}} + o_{\mathrm{C}} \iota_{\mathrm{A}})\,\epsilon_{\dot{\mathrm{B}}\dot{\mathrm{D}}} . \tag{51}$$

The real tetrad gives rise to the following six skew-symmetric tensors, which are quite relevant in the study of the movement of classical charged particles [34-37, 49]:

$$\mathrm{W}_{(j)\,\mu\nu} = \mathrm{e}_{(0)\,\mu} \ \mathrm{x} \ \mathrm{e}_{(j)\,\nu} , \quad j = 1, 2, 3, \qquad \mathrm{W}_{(4)\,\mu\nu} = \mathrm{e}_{(1)\,\mu} \ \mathrm{x} \ \mathrm{e}_{(2)\,\nu} , \tag{52}$$
$$\mathrm{W}_{(5)\,\mu\nu} = \mathrm{e}_{(1)\,\mu} \ \mathrm{x} \ \mathrm{e}_{(3)\,\nu} , \qquad \mathrm{W}_{(6)\,\mu\nu} = \mathrm{e}_{(2)\,\mu} \ \mathrm{x} \ \mathrm{e}_{(3)\,\nu} ,$$

and from (39) (without sum over r):

$$\mathrm{W}_{(r)\,\mu\nu}\,\mathrm{W}_{(r)}{}^{\mu\nu} = -2 \text{ or } 2 \quad \text{for} \quad r = 1, 2, 3 \quad \text{or} \quad r = 4, 5, 6, \text{ respectively,}$$
$$\mathrm{W}_{(j)\,\mu\nu}\,\mathrm{W}_{(k)}{}^{\mu\nu} = 0, \qquad j \neq k\,, \qquad j,k = 1,2,\dots,6\,. \tag{53}$$

The concept of dual tensor [22, 50-53]:

$$^{+}\mathrm{F}_{\mu\nu} = \frac{1}{2}\,\eta_{\mu\nu\alpha\beta}\,\mathrm{F}^{\alpha\beta}, \qquad \mathrm{F}_{\mu\nu} = -\mathrm{F}_{\nu\mu}\,, \tag{54}$$

together with (52), implies the connection:

$$^{+}\mathrm{W}_{(n)\,\mu\nu} = -(-1)^{n}\,\mathrm{W}_{(7-n)\,\mu\nu}\,, \qquad n = 1,2,\dots,6\,, \tag{55}$$

namely, $^{+}\mathrm{W}_{(2)\,\alpha\beta} = -\mathrm{W}_{(5)\,\alpha\beta}$, $^{+}\mathrm{W}_{(3)\,\alpha\beta} = \mathrm{W}_{(4)\,\alpha\beta}$, etc., with importance in the study of the algebraic composition of the Faraday tensor.

By means of (46) and (51) it is obtained the spinorial version of (52):

$$\begin{aligned}
\mathrm{W}_{(1)\,\mathrm{AC\dot{B}\dot{D}}} &= \tfrac{1}{2}\left[(o_{\mathrm{A}}o_{\mathrm{C}} - \iota_{\mathrm{A}}\iota_{\mathrm{C}})\,\epsilon_{\mathrm{\dot{B}\dot{D}}} + \epsilon_{\mathrm{AC}}\,(o_{\mathrm{\dot{B}}}o_{\mathrm{\dot{D}}} - \iota_{\mathrm{\dot{B}}}\iota_{\mathrm{\dot{D}}})\right],\\
\mathrm{W}_{(2)\,\mathrm{AC\dot{B}\dot{D}}} &= \tfrac{i}{2}\left[(o_{\mathrm{A}}o_{\mathrm{C}} + \iota_{\mathrm{A}}\iota_{\mathrm{C}})\,\epsilon_{\mathrm{\dot{B}\dot{D}}} - \epsilon_{\mathrm{AC}}\,(o_{\mathrm{\dot{B}}}o_{\mathrm{\dot{D}}} + \iota_{\mathrm{\dot{B}}}\iota_{\mathrm{\dot{D}}})\right],\\
\mathrm{W}_{(3)\,\mathrm{AC\dot{B}\dot{D}}} &= -\tfrac{1}{2}\left[(o_{\mathrm{A}}\iota_{\mathrm{C}} + o_{\mathrm{C}}\iota_{\mathrm{A}})\,\epsilon_{\mathrm{\dot{B}\dot{D}}} + \epsilon_{\mathrm{AC}}\,(o_{\mathrm{\dot{B}}}\iota_{\mathrm{\dot{D}}} + o_{\mathrm{\dot{D}}}\iota_{\mathrm{\dot{B}}})\right],\\
\mathrm{W}_{(4)\,\mathrm{AC\dot{B}\dot{D}}} &= \tfrac{i}{2}\left[(o_{\mathrm{A}}\iota_{\mathrm{C}} + o_{\mathrm{C}}\iota_{\mathrm{A}})\,\epsilon_{\mathrm{\dot{B}\dot{D}}} - \epsilon_{\mathrm{AC}}\,(o_{\mathrm{\dot{B}}}\iota_{\mathrm{\dot{D}}} + o_{\mathrm{\dot{D}}}\iota_{\mathrm{\dot{B}}})\right],\\
\mathrm{W}_{(5)\,\mathrm{AC\dot{B}\dot{D}}} &= -\tfrac{1}{2}\left[(o_{\mathrm{A}}o_{\mathrm{C}} + \iota_{\mathrm{A}}\iota_{\mathrm{C}})\,\epsilon_{\mathrm{\dot{B}\dot{D}}} + \epsilon_{\mathrm{AC}}\,(o_{\mathrm{\dot{B}}}o_{\mathrm{\dot{D}}} + \iota_{\mathrm{\dot{B}}}\iota_{\mathrm{\dot{D}}})\right],\\
\mathrm{W}_{(6)\,\mathrm{AC\dot{B}\dot{D}}} &= \tfrac{i}{2}\left[-(o_{\mathrm{A}}o_{\mathrm{C}} - \iota_{\mathrm{A}}\iota_{\mathrm{C}})\,\epsilon_{\mathrm{\dot{B}\dot{D}}} + \epsilon_{\mathrm{AC}}\,(o_{\mathrm{\dot{B}}}o_{\mathrm{\dot{D}}} - \iota_{\mathrm{\dot{B}}}\iota_{\mathrm{\dot{D}}})\right],
\end{aligned} \tag{56}$$

where it is verified the property $\overline{\mathrm{W}_{(r)\,\mathrm{AC\dot{B}\dot{D}}}} = \mathrm{W}_{(r)\,\mathrm{BD\dot{A}\dot{C}}}$ because the $\mathrm{W}_{(r)\,\mu\nu}$ are real. The relations given by (56) motivate the following comment of Rindler [20]:

‘Every spinor can be written as a linear combination of symmetric spinors multiplied by ϵ_{AB} or/and $\epsilon_{\mathrm{\dot{A}\dot{B}}}$’. (57)

In the structure of (56) we can observe the repetition of different kinds of terms, then it is natural to introduce the spinors:

$$\mathrm{V}_{\mathrm{AC\dot{B}\dot{D}}} = o_{\mathrm{A}}o_{\mathrm{C}}\,\epsilon_{\mathrm{\dot{B}\dot{D}}}\,, \qquad \mathrm{U}_{\mathrm{AC\dot{B}\dot{D}}} = \iota_{\mathrm{A}}\iota_{\mathrm{C}}\,\epsilon_{\mathrm{\dot{B}\dot{D}}}\,, \qquad \mathrm{M}_{\mathrm{AC\dot{B}\dot{D}}} = -(o_{\mathrm{A}}\iota_{\mathrm{C}} + o_{\mathrm{C}}\iota_{\mathrm{A}})\,\epsilon_{\mathrm{\dot{B}\dot{D}}}\,, \tag{58}$$

and the prescription (24), together with (44), (45) and (50), gives their tensorial counterpart:

$$\mathrm{V}_{\mu\nu} = l_{\mu} \times m_{\nu}\,, \qquad \mathrm{U}_{\mu\nu} = \bar{m}_{\mu} \times n_{\nu}\,, \qquad \mathrm{M}_{\mu\nu} = m_{\mu} \times \bar{m}_{\nu} + n_{\mu} \times l_{\nu}\,, \tag{59}$$

with importance in the formalism of NP [33, 45-47].

The Levi-Civita tensor admits a representation in terms of the real and NP tetrads:

$$\eta_{\mu\nu\alpha\beta} = -\begin{vmatrix} e_{(0)\mu} & e_{(0)\nu} & e_{(0)\alpha} & e_{(0)\beta} \\ e_{(1)\mu} & e_{(1)\nu} & e_{(1)\alpha} & e_{(1)\beta} \\ e_{(2)\mu} & e_{(2)\nu} & e_{(2)\alpha} & e_{(2)\beta} \\ e_{(3)\mu} & e_{(3)\nu} & e_{(3)\alpha} & e_{(3)\beta} \end{vmatrix} = -i \begin{vmatrix} l_\mu & l_\nu & l_\alpha & l_\beta \\ n_\mu & n_\nu & n_\alpha & n_\beta \\ m_\mu & m_\nu & m_\alpha & m_\beta \\ \overline{m}_\mu & \overline{m}_\nu & \overline{m}_\alpha & \overline{m}_\beta \end{vmatrix}, \tag{60}$$

where the positive orientation indicated by (40) and (43) is respected. With (52), (54) and (60) it is easy to prove (55); besides, (60) leads to the relations:

$$\eta_{\mu\nu\alpha\beta}\, l^\alpha m^\beta = -i\, l_\mu \text{ x } m_\nu\,, \qquad \eta_{\mu\nu\alpha\beta}\, m^\alpha \overline{m}^\beta = i\, l_\mu \text{ x } n_\nu\,, \tag{61}$$

$$\eta_{\mu\nu\alpha\beta}\, n^\alpha m^\beta = \; i\, n_\mu \text{ x } m_\nu\,, \qquad \eta_{\mu\nu\alpha\beta}\, l^\alpha n^\beta = i\, m_\mu \text{ x } \overline{m}_\nu\,,$$

which imply the self-dual character of (59):

$$^{+}V_{\mu\nu} = -i\, V_{\mu\nu}\,, \qquad ^{+}U_{\mu\nu} = -i\, U_{\mu\nu}\,, \qquad ^{+}M_{\mu\nu} = -i\, M_{\mu\nu}\,. \tag{62}$$

Projecting (60) onto the Infeld-van der Waerden symbols [54], and employing (44), (45) and (50), we obtain the corresponding spinor [39]:

$$\eta_{\mathrm{ACEG\dot{B}\dot{D}\dot{F}\dot{H}}} = i\,(\epsilon_{\mathrm{AE}}\,\epsilon_{\mathrm{CG}}\,\epsilon_{\mathrm{\dot{B}\dot{H}}}\,\epsilon_{\mathrm{\dot{D}\dot{F}}} - \epsilon_{\mathrm{AG}}\,\epsilon_{\mathrm{CE}}\,\epsilon_{\mathrm{\dot{B}\dot{F}}}\,\epsilon_{\mathrm{\dot{D}\dot{H}}}). \tag{63}$$

Let $F_{\mu\nu}$ be an arbitrary tensor, then the prescription (16) gives its associated spinor:

$$F_{\mathrm{AC\dot{B}\dot{D}}} = \sigma^{\mu}{}_{\mathrm{A\dot{B}}}\, F_{\mu\nu}\, \sigma^{\nu}{}_{\mathrm{C\dot{D}}}\,, \tag{64}$$

from where:

$$\overline{F_{\mathrm{AC\dot{B}\dot{D}}}} = F_{\mathrm{BD\dot{A}\dot{C}}} \tag{65}$$

because $F_{\alpha\beta}$ is real, $F_{\mathrm{AC\dot{B}\dot{D}}} = -F_{\mathrm{CA\dot{D}\dot{B}}}$ because $F_{\mu\nu}$ is skew-symmetric,

these results together with (54) and (63) lead to the spinor associated with the dual tensor:

$$^{+}F_{\mathrm{AC\dot{B}\dot{D}}} = i\, F_{\mathrm{AC\dot{D}\dot{B}}}\,, \tag{66}$$

which also verifies the symmetries (65).

Penrose [40] asseverates that the 2-spinors formalism is not only simpler when it comes to establish properties of conformal invariance, but does also provide a more systematical overview when it comes to understand the propagation of massless fields. Then, Sec. 4 is devoted to the study of some spinorial aspects of the electromagnetic field.

(*Continued on Part II*)