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Solution of the Kepler Problem in the Framework of Lorentz-invariant Gravitation Theory

Alexander G. Kyriakos*

Saint-Petersburg State Institute of Technology, St.-Petersburg, Russia

Abstract

In present paper, based on previous results of Lorentz-invariant Gravitation Theory (LIGT), we consider the solution of the Kepler problem, i.e., the solution of the problem of motion of two bodies in a centrally symmetric gravitational field of a stationary source. It is shown that this solution coincides with that obtained in GR.

Key words: Lorentz-invariant theory, Kepler problem, Schwarzschild-Droste metric.

Abbreviations:	**Designation:**
GTR or GR - General Theory of Relativity	γ_N - Newton's constant of gravitation
LIGT- Lorentz-invariant gravitation theory	γ_L - Lorentz factor (L-factor)
HJE - Hamilton-Jacobi equation	m - mass of particle
L-transformation - Lorentz transformation	M_S - mass of the star (Sun)
L-invariant - Lorentz-invariant	M - angular momentum

1.0. Introduction. Motion in the central force field

The central force problem is called such a problem, in which the ordinary potential function depends only on the magnitude of the relative distance between the particles (r): $U(\vec{r}) = U(r)$.

The central force motion is one of the oldest and widely studied problems in classical mechanics. Several familiar force-laws in nature, e.g., Newton's law of gravitation, Coulomb's law, van-der Waals force, Yukawa interaction, and Hooke's law are all examples of central forces.

The motion of a particle in a central field, is both the movement of the planet in the solar system and the motion of an electron in an atom.

Further, as a review, we present a summary of the formulation and solution of the Kepler problem in various cases (see paragraphs 2.0 to 4.0 inclusively)

(Note that we will continue to consider the terms "relativistic" and "Lorentz-invariant (L-invariant)" as equivalent).

* Correspondence: AlexanderG.Kyriakos, Saint-Petersburg State Institute of Technology, St.-Petersburg, Russia.
Present address: Athens, Greece, E-mail: a.g.kyriak@hotmail.com

2.0 The non-Lorentz-invariant case of motion in a central field

2.1. General approach (Murayama, 2007; Ray and Shamanna, 2004).

When a particle is moving in a central potential $U(\vec{r})=U(r)$, a function only of the radius r, the Hamilton–Jacobi equation can be solved by using the spherical coordinates. The Lagrangian is

$$L=\frac{1}{2}m\dot{\vec{r}}^{2}-U(r), \tag{2.1}$$

going to the spherical coordinates, it becomes

$$L=\frac{1}{2}m\left(\dot{r}^{2}+r^{2}\dot{\theta}^{2}+r^{2}\dot{\varphi}^{2}\sin^{2}\theta\right)-U(r), \tag{2.2}$$

The canonical momenta are defined as

$$p_r=m\dot{r},\quad p_\theta=mr^2\dot{\theta},\quad p_\varphi=mr^2\dot{\varphi}\sin^2\theta, \tag{2.3}$$

It is easy to verify that the generalized momenta p_r, p_θ, p_φ are related to the momentum $\vec{p}=m\vec{\upsilon}$ and angular momentum $\vec{M}=\left[\vec{r}\times\vec{p}\right]$ by means of relationships $p_r=(\vec{p})_r=\vec{p}\vec{r}^{\,0}$, $\vec{\mathrm{p}}^{2}=p_r^2+\dfrac{M^2}{r^2}$, $p_\varphi=M_z$, $p_\theta^2+\dfrac{p_\varphi^2}{\sin^2\theta}=\vec{M}^2$.

Following the definition, we find the Hamiltonian

$$H=\frac{p_r^2}{2m}+\frac{p_\theta^2}{2mr^2}+\frac{p_\varphi^2}{2mr^2\sin^2\theta}+U(r), \tag{2.4}$$

Then the non-L-invariant Hamilton–Jacobi equation is found to be

$$\frac{\partial S}{\partial t}+\frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2+\frac{1}{2mr^2}\left(\frac{\partial S}{\partial\theta}\right)^2+\frac{1}{2mr^2\sin^2\theta}\left(\frac{\partial S}{\partial\varphi}\right)^2+U(r)=0, \tag{2.5}$$

It can be simplified using the method of separation of variables. Separation of variables is done in the following simple manner:

$$S(t,r,\theta,\varphi)=S_t(t)+S_r(r)+S_\theta(\theta)+S_\varphi(\varphi), \tag{2.6}$$

Then the Hamilton–Jacobi equation becomes

$$\frac{\partial S_t}{\partial t}+\frac{1}{2m}\left(\frac{\partial S_r}{\partial r}\right)^2+\frac{1}{2mr^2}\left(\frac{\partial S_\theta}{\partial\theta}\right)^2+\frac{1}{2mr^2\sin^2\theta}\left(\frac{\partial S_\varphi}{\partial\varphi}\right)^2+U(r)=0, \tag{2.5'}$$

Because there are no explicit t- and φ-dependence in the equation, we conclude $\partial S_t/\partial t$, $\partial S_\varphi/\partial\varphi$ must be constant. We set

$$\frac{\partial S_t}{\partial t}=-\varepsilon,\quad \frac{\partial S_\varphi}{\partial\varphi}=M_z, \tag{2.7}$$

It can be shown that they have the meaning of the energy ε and the z-component of the orbital angular momentum M_z.

The full solution for the motion in a central field is most easily obtained from the laws of conservation of energy and momentum.

2.2. Radial motion (Parshin and Zegria, 1998. Lect 9th; Kotkin et all. 2007).

When a particle moves in a central field, the force $\vec{F} = -gradU$ is directed along the radius vector, or against it; therefore energy is not the only thing saved,

$$\varepsilon = \frac{1}{2}m\upsilon^2 + U(r), \tag{2.8}$$

the angular momentum is saved as well:

$$\vec{M} = m[\vec{r} \times \vec{\upsilon}], \tag{2.9}$$

since $d\vec{M}/dt = [\vec{r} \times \vec{F}] = 0$

From the equation (2.9) it follows that the orbit of a particle is in a plane, which is perpendicular to the constant vector $\vec{M}$; let it be *xy*- plane. Introducing in this plane the polar coordinates r and φ, we obtain:

$$\varepsilon = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r\dot{\varphi})^2 + U(r), \tag{2.10}$$

$$\vec{M} = (0,0,M), \quad M = M_z = mr^2\dot{\varphi}, \tag{2.11}$$

Using (2.11), we eliminate $\dot{\varphi}$ from (2.10):

$$\varepsilon = \frac{1}{2}m\dot{r}^2 + U_{ef}(r), \tag{2.12}$$

where $U_{ef}(r) = U(r) + \frac{M^2}{2mr^2}$. Thus, the radial motion is reduced to one-dimensional movement in the field with an effective potential energy $U_{ef}(r)$ comprising a centrifugal energy $M^2/2mr^2$. From (2.12) we obtain:

$$dt = \pm\sqrt{\frac{m}{2}}\frac{dr}{\sqrt{\varepsilon - U_{ef}(r)}}, \tag{2.13}$$

from which, by means of the integration, we obtain the dependence $t = t(r)$. In this case, the trajectory is determined by equation $r = r(\varphi)$ To derive the equation of the trajectory we will use (2.11) in the form $d\varphi = \frac{M}{mr^2}dt$. Substituting dt, we obtain from (2.13):

$$\varphi = \pm\frac{M}{\sqrt{2m}}\int_{r_0}^{r}\frac{dr}{r^2\sqrt{\varepsilon - U_{ef}(r)}} + \varphi_0, \tag{2.14}$$

2.3. Kepler problem

Let us consider the motion of a particle in a potential field:

$$U(r) = -\frac{\alpha}{r}, \tag{2.15}$$

Here $\alpha = \gamma_N m M_S$, for the motion of the planet of mass m in a gravitational field of the star, such as the Sun, with the mass M_S; or $\alpha = q \cdot Q$ for the movement of the body with the mass m and charge q in the electric field of the charge Q belonging to the body of mass M_S (for the motion of an electron in the electric field of the proton $q = Q = e$, $\alpha = e^2$, where e is the electron charge, which corresponds to hydrogen atom for high, so-called, Rydberg states). When moving in a gravitational field or in Coulomb attractive field $U(r) = -\alpha/r$ (in the Coulomb field of repulsion $U(r) = \alpha/r$).

2.3.1. *Trajectories*

The effective potential energy for this field is:

$$U_{ef}(r) = -\frac{\alpha}{r} + \frac{M^2}{2mr^2}, \tag{2.16}$$

As a result of integrating of the equations of motion (2.14), we obtain the trajectory equation in the form:

$$r = \frac{p}{1 + e\cos\varphi}, \tag{2.17}$$

where $e = \sqrt{1 + 2\varepsilon M^2 / m\alpha^2}$ is eccentricity, and $p = M^2/m\alpha$ is the parameter of the orbit ($p/r = u$ is dimensionless).

The equation (2.17) defines the known curves: an ellipse (e < 1), a parabola (e = 1) and a hyperbola (e > 1), which are obtained depending on $\varepsilon < 0$ (bound state), $\varepsilon = 0$, and $\varepsilon > 0$.

2.4. Motion in the central field of the form $U(r) = -\frac{\alpha}{r} + \frac{\beta}{r^2}$

Effective potential energy for this field is:

$$U_{ef}(r) = -\frac{\alpha}{r} + \frac{M^2}{2mr^2} + \frac{\beta}{r^2}, \tag{2.18}$$

or

$$U_{ef}(r) = -\frac{\alpha}{r} + \frac{\tilde{M}^2}{2mr^2}, \quad \tilde{M} = \sqrt{M^2 + 2m\beta}, \tag{2.18'}$$

The trajectory is determined from the equation (2.14):

$$\varphi = \pm\int \frac{M}{r^2} \frac{dr}{\sqrt{2m\varepsilon + \frac{2m\alpha}{r} - \frac{\tilde{M}}{r^2}}} + \varphi_0, \tag{2.19}$$

In this case we obtain the trajectory equation in form:

$$r = \frac{\tilde{p}}{1 + \tilde{e}\cos(\gamma\varphi)}, \tag{2.20}$$

where we introduced the notations: $\gamma = \tilde{M}/M = \sqrt{1 + 2m\beta/M^2}$, $\tilde{p} = \tilde{M}^2/m\alpha$, $\tilde{e} = \sqrt{1 + 2\varepsilon\tilde{M}^2/m\alpha^2}$.

For arbitrary values of the angular momentum, the trajectory is an unclosed curve (see. Fig. 2.1).

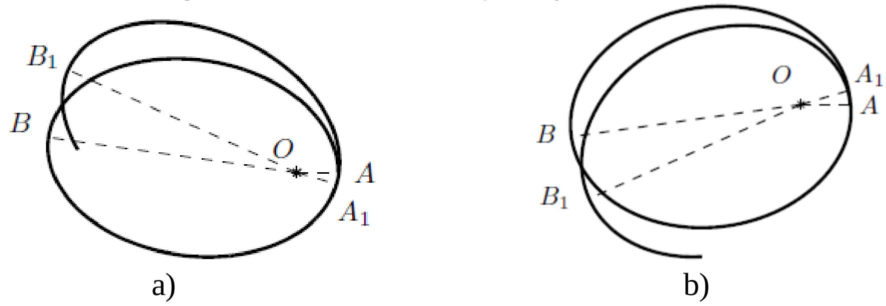

Fig. 2.1 . The trajectory, which is described by (2.20) in the case of $\varepsilon < 0$: a) when $\beta > 0$ and b) when $\beta < 0$.

On Fig. 2.1 the points A and B correspond to the motion of the perihelion (perihelion moves clockwise when $\beta > 0$ and anticlockwise when $\beta < 0$).

The above example is typical for motion in a central field, which we divided into two movements: by angle φ and by radius r . At finite motion both motions are periodic, but periods T_φ and T_r , in general, are incommensurable, and therefore, the trajectory of finite motion, generally speaking, is not closed.

Often the problems of motion of bodies under the action of central forces are reduced to the solution of the Hamilton-Jacobi equation (HJE).

The Hamilton-Jacobi method of integration (Sommerfeld, 1952a). leads directly to the solution of the planetary motion problem. From other hand the same method is made to order for the requirements of atomic physics and leads in a natural way to the (older) quantum theory.

3.0 The Lorentz-invariant motion in a central field

L-invariant HJE of the particle motion in a central field of energy U has a well-known form:

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t} + U\right)^2 - (grad\, S)^2 = m^2c^2 \ .$$

In spite of the Kepler problem (Sommerfeld, 1952b) of the hydrogen atom the equations lead to an ellipse with precessing perihelion instead of to a closed ellipse as a consequence of the relativistic variation of mass. (Kotkin et all. 2007).

Generalization of the formulas of the movement theory in the central field on the relativistic case is reached by the transition to the Lorentz-invariant (relativistic) Lagrangian (Kotkin et all. 2007).:

$$L(\vec{r},\vec{\upsilon},t) = -mc^2\sqrt{1-\frac{\vec{\upsilon}^2}{c^2}} - e\varphi + \frac{e}{c}\vec{A}\vec{\upsilon}\,, \tag{3.1}$$

and Hamiltonian:

$$H(\vec{p},\vec{r},t) = \sqrt{\left(\vec{p}-\frac{e}{c}A(\vec{r},t)\right)^2 c^2 + m^2c^4} + e\varphi(\vec{r},t), \tag{3.2}$$

In our case, when the motion of a relativistic particle is in the field $U(r) = -\frac{\alpha}{r}$, and the vector potential is zero, $\vec{A} = 0$, the Hamiltonian of a relativistic particle is given by:

$$H(\vec{p},\vec{r},t) = \sqrt{\vec{p}^2c^2 + m^2c^4} - \frac{\alpha}{r}\,, \tag{3.3}$$

where the relativistic momentum is $\vec{p} = m\vec{\upsilon}\Big/\sqrt{1-(\vec{\upsilon}/c)^2}$. When moving in a central field the relativistic energy ε' and angular momentum $\vec{M}'$ are conserved:

$$\varepsilon' = \sqrt{\vec{p}^2c^2 + m^2c^4} - \frac{\alpha}{r}, \quad \vec{M}' = [\vec{r}\times\vec{p}] = m[\vec{r}\times\vec{\upsilon}]\Big/\sqrt{1-(\vec{\upsilon}/c)^2}\,, \tag{3.4}$$

3.1. Motion of electron in the coulomb attractive field

Our problem (Landau and Lifshitz , 1971) becomes the study of the motion of a charge $-e$ in a centrally symmetric electric field with potential $\varphi = e'/r$. The total energy of the particle is equal to

$$\varepsilon' = c\sqrt{p^2 + m^2c^2} - \frac{\alpha}{r}, \tag{3.5}$$

where $\alpha = ee'$. If we use polar coordinates in the plane of motion of the particle, then as we know from mechanics,

$$p'^2 = \frac{M'^2}{r^2} + p'^2_r\,, \tag{3.6}$$

where $\vec{p}'$ is relativistic momentum, $\vec{p}' = m\vec{\upsilon}\Big/\sqrt{1-(\vec{\upsilon}/c)^2}$, p'_r is the radial component of the momentum, and M is the constant angular momentum of the particle
$\vec{M}' = [\vec{r}\times\vec{p}] = m[\vec{r}\times\vec{\upsilon}]\Big/\sqrt{1-(\vec{\upsilon}/c)^2}$; the corresponding general momenta are
$p'_r = \partial L/\partial\dot{r} = m\dot{r}\Big/\sqrt{1-(\vec{\upsilon}/c)^2}$, $p'_\varphi = \partial L/\partial\dot{\varphi} = mr^2\dot{\varphi}\Big/\sqrt{1-(\vec{\upsilon}/c)^2}$.
Из (3.6), учитывая (3.5) нетрудно получить:

$$p'_r(r) = \sqrt{\frac{1}{c^2}\left(\varepsilon' - \frac{\alpha}{r}\right)^2 - \frac{M'^2}{r^2} - m^2c^2}\,, \tag{3.7}$$

Then

$$\varepsilon' = c\sqrt{\frac{M'^2}{r^2} + p'^2_r + m^2c^2} - \frac{\alpha}{r}, \tag{3.8}$$

A complete determination of the motion of a charge in a Coulomb field starts most conveniently from the Hamilton- Jacobi equation. We choose polar coordinates r, φ in the plane of the motion. In this case the L-invariant Hamilton-Jacobi equation has the form:

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t} + \frac{\alpha}{r}\right)^2 - \left(\frac{\partial S}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 = m^2c^2, \tag{3.9}$$

We seek an S of the form

$$S = -\varepsilon' t + M'\varphi + f(r), \tag{3.10}$$

where ε, M are the constant energy and angular momentum of the moving particle, which play the role of parameters, and $f(r) = \int p'_r(r)dr = \int \sqrt{\frac{1}{c^2}\left(\varepsilon' - \frac{\alpha}{r}\right)^2 - \frac{M'^2}{r^2} - m^2c^2} \cdot dr$.

The trajectory is determined by the equation $\frac{\partial S}{\partial M'} = const$ or

$$\varphi = \int \frac{M'}{r^2} \frac{dr}{p'_r(r)}, \tag{3.11}$$

The integration constant is contained in the arbitrary choice of the reference line for measurement of the angle φ.

Let us compare the equation (3.11) (Kotkin et all., 2007) with the equation (2.19) for the trajectory of motion of a non-relativistic particle in the field $U(r) = -\frac{\alpha}{r} + \frac{\beta}{r^2}$. It can be seen that equation (3.11) coincides with the equation (2.19) by replacing

$$m \to \frac{\varepsilon'}{c^2},\ \varepsilon \to \frac{\varepsilon'^2 - m^2c^4}{2\varepsilon'},\ \beta \to \frac{\alpha^2}{2\varepsilon'}, \tag{3.12}$$

In other words, the equation (3.11) corresponds to the motion of a non-relativistic particle in a Coulomb field $U(r) = -\frac{\alpha}{r}$ with a perturbation in the form of the additional central attraction field

$$\delta U(r) = -\frac{\alpha^2}{2\varepsilon' r^2}, \tag{3.13}$$

For $M' > \alpha / c$ this allows us to write immediately the answer for our case in a form similar to (2.20):

$$r = \frac{\tilde{p}}{1 + \tilde{e}\cos(\gamma\varphi)}, \tag{3.14}$$

where we introduced the notation

$$\gamma = \sqrt{1 - \frac{\alpha^2}{c^2M'^2}},\ \tilde{p} = \frac{c^2M'^2 - \alpha^2}{\varepsilon'\alpha},\ \tilde{e} = \sqrt{1 + \frac{(\varepsilon'^2 - m^2c^4)(c^2M'^2 - \alpha^2)}{\varepsilon'^2\alpha^2}}, \tag{3.15}$$

For the case $\varepsilon' < mc^2$ (when $\tilde{e} < 1$), the trajectory corresponds to the finite motion of the particle and, generally speaking, is not closed (see Fig. 2.1). During one radial particle oscillation its polar angle is changed to

$$\Delta\varphi = \frac{2\pi}{\gamma}, \tag{3.16}$$

therefore the perihelion is shifted on angle

$$\delta\varphi = \frac{2\pi}{\gamma} - 2\pi, \tag{3.17}$$

For the motion of the planets in the solar system non-relativistic limit of this expression can be used (considering $\upsilon << c$ or $mc^2 - \varepsilon' \approx \alpha/2a << mc^2$)

$$\delta\varphi \approx \pi \frac{\alpha/a}{mc^2\left(1-e^2\right)}, \tag{3.18}$$

where a and e are major semi-axis and eccentricity of the planet. For Mercury the numerical value of the perihelion is equal

$$\delta\varphi \approx 7{,}2'' \text{ per a century}, \tag{3.19}$$

The observed precession of the perihelion of Mercury (after excluding the influence of other planets) is equal to:

$$\delta\varphi_{\text{observat}} = (43,\ 11 \pm 0,\ 45)'' \text{ per a century}, \tag{3.20}$$

This value $\delta\varphi_{\text{observat}}$ is in obvious contradiction with the prediction (3.20) of special theory of relativity, but is in excellent agreement with the predictions of general relativity:

$$\delta\varphi_{\text{OTO}} = 6\,\delta\varphi \approx 43'' \text{ per a century}, \tag{3.21}$$

Thus, taking into account the relativistic electron mass change, we came to the precession of the orbit of an electron in a hydrogen atom. Direct (mathematical) applying of this effect to gravity, gives a precession value, much lower than the one obtained by measurement. In the future, we will show that in framework of LITG, the inclusion of relativistic effects gives the desired result.

3.2. The transition to the limiting case of non-Lorentz-invariant equation of motion in a central field

Let us use the general form of L-invariant HJE of the particle motion in a central field of energy U (Landau and Lifshitz, 1971).:

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t} + U\right)^2 - \left(grad\, S\right)^2 = m^2c^2, \tag{3.22}$$

The transition to the limiting case of classical (non-relativistic) mechanics in equation (3.22) is made, as follows. First of all we must notice that the energy of a particle in relativistic mechanics contains the term mc^2, which it does not in classical mechanics. Inasmuch as the action S is related, to the energy by $\varepsilon = -\partial S/\partial t$, in making the transition to classical mechanics we must in place of S

substitute a new action S' according to the relation: $S = S' - mc^2 t$. Substituting this in (3.22), we find

$$\left(\frac{\partial S'}{\partial t} - mc^2 + U\right)^2 - c^2 (grad\, S)^2 = m^2 c^4 \ , \tag{3.23}$$

Raising the first term on the left in the square, we get:

$$\left(\frac{\partial S'}{\partial t}\right)^2 + m^2c^4 + U^2 - 2mc^2 \frac{\partial S'}{\partial t} + 2U \frac{\partial S'}{\partial t} - 2mc^2 U - c^2 (grad\, S)^2 = m^2 c^4 \ , \tag{3.24}$$

Reducing the equation to m^2c^4 and dividing it by mc^2, we get

$$\frac{1}{2mc^2}\left(\frac{\partial S'}{\partial t}\right)^2 + \frac{U^2}{2mc^2} - \frac{\partial S'}{\partial t} + \frac{U}{mc^2}\frac{\partial S'}{\partial t} - U - \frac{1}{2m}(grad\, S)^2 = 0 \ , \tag{3.25}$$

In spherical coordinates, this equation will have the form:

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{2mr^2 \sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2 + U(r) = 0 , \tag{3.27}$$

4.0. On the Schwarzschild-Droste solution of GR equation.

We will test the correctness of LIGT equations by the existing solutions for general relativity equations, which were verified experimentally. The first of these is the Schwarzschild -Droste solution.

Soon after the GR creation K. Schwarzschild (Schwarzschild, 1916) has obtained the exact static spherically symmetric solution of the vacuum Einstein equations - the Schwarzschild metric. Johannes Droste (a disciple of Lorentz) in 1916 independently produced the same solution as Schwarzschild, using a simpler, more direct derivation (Droste, 1917) Somewhat later, the solution to this problem was obtained by Hilbert (Hilbert, 1917). All these solutions are different from each other, but give the same results (Kox, 1992).

Schwarzschild-Droste solution plays an important role in physical applications of general relativity. In fact, the main observational evidence of general relativity, was obtained on the basis of this decision.

The space-time metric of Schwarzschild- Droste has the form (Landau and Lifshitz,, 1971, § 100, p. 301):

$$ds^2 = \left(1 - \frac{r_s}{r}\right)c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{r_s}{r}\right)} - r^2\left(\sin^2\theta d\varphi^2 + d\theta^2\right), \tag{4.1}$$

where the quantity $r_s = \dfrac{2\gamma_N M_s}{c^2}$ has the dimensions of length and it is called the gravitational radius of the body, and the mass of the central body we denote by M_s.

This solution of the Einstein equations completely determines the gravitational field in vacuum produced by any centrally-symmetric distribution of masses. We emphasize that this solution is valid not only for masses at rest, but also when they are moving, so long as the motion has the required symmetry. We note that the metric (4.1) depends only on the total mass of the gravitating body, just as in the analogous problem in Newtonian theory. The spatial metric is determined by the expression for the element of spatial distance:

$$dl^2 = \frac{dr^2}{\left(1-\dfrac{r_s}{r}\right)} + r^2\left(\sin^2\theta d\varphi^2 + d\theta^2\right), \tag{4.2}$$

The geometrical meaning of the coordinate r is determined by the fact that in the metric (6.2) the circumference of a circle with its center at the center of the field is $2\pi r$. But the distance between two points r_2 and r_2 along the same radius is given by the integral:

$$\int_{r_1}^{r_2} \frac{dr}{\sqrt{1-r_s/r}} > r_2 - r_1, \tag{4.3}$$

4.1. Motion in a centrally symmetric gravitational field)

To determine the trajectory of the particle, we use the Hamilton-Jacobi equation (Landau and Lifshitz, 1971) :

$$g^{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k} - m^2c^2 = 0, \tag{4.4}$$

where m is the mass of the particle. Using the gik given in the expression (4.1), we find the following equation:

$$\frac{1}{1-\dfrac{r_s}{r}}\left(\frac{\partial S}{\partial t}\right)^2 - c^2\left(1-\frac{r_s}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2 - \frac{c^2}{r^2}\left[\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2\right] = m^2c^4 \ , \tag{4.5}$$

Let us consider the motion of a particle in a centrally symmetric gravitational field. As in every centrally symmetric field, the motion occurs in a single "plane" passing through the origin; we choose this plane as the plane $\theta = \pi/2$.

$$\frac{1}{1-\dfrac{r_s}{r}}\left(\frac{\partial S}{c\partial t}\right)^2 - \left(1-\frac{r_s}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 = m^2c^2 \ , \tag{4.6}$$

By the general procedure for solving the Hamilton-Jacobi equation, we look for an S in the form $S = -\varepsilon_0 t + M\varphi + S_r(r)$ with constant energy ε_0 and angular momentum M. Substituting this expression in (4.6), we find the equation:

$$\frac{1}{1-\frac{r_s}{r}}\frac{\varepsilon_0^2}{c^2}-\left(1-\frac{r_s}{r}\right)\left(\frac{\partial S_r}{\partial r}\right)^2-\frac{M^2}{r^2}=m^2c^2, \tag{4.7}$$

The trajectory itself is determined by the equation $\partial S/\partial M=\varphi+\partial S_r/\partial M=const$, so that the orbits of a test particle of infinitesimal mass m about the central mass M is given by the equation of motion, which can be converted into an equation for the orbit

$$\left(\frac{dr}{d\varphi}\right)^2=\frac{r^4}{b^2}-\left(1-\frac{r_s}{r}\right)\left(\frac{r^4}{a^2}+r^2\right), \tag{4.8}$$

where, for brevity, two length-scales, a and b, have been introduced. They are constants of the motion and depend on the initial conditions (position and velocity) of the test particle. Hence, the solution of the orbit equation is

$$\varphi=\int\frac{dr}{r^2\sqrt{\frac{1}{1/b^2-(1-r_s/r)(1/a^2+1/r^2)}}}, \tag{4.9}$$

4.1.1. *Effective radial potential energy*

It can be shown that the motion of a particle in the Schwarzschild field is equivalent to the motion of a nonrelativistic particle with energy $\left(\frac{\varepsilon_0}{2mc^2}-\frac{1}{2}mc^2\right)$ in a one-dimensional effective potential

$$U(r)=-\frac{\gamma_N mM_s}{r}+\frac{M^2}{2mr^2}-\frac{\gamma_N M_s M^2}{c^2mr^3}, \tag{4.10}$$

which can be rewritten through the parameters of length a and b as:

$$U(r)=\frac{mc^2}{2}\left(-\frac{r_s}{r}+\frac{a^2}{r^2}-\frac{r_s a^2}{r^3}\right), \tag{4.11}$$

The first two terms are well-known classical energies: the first is the attractive Newtonian gravitational potential energy and the second corresponds to the repulsive "centrifugal" potential energy; however, the third term is an attractive energy, unique to classical mechanics. As it is known, this inverse-cubic energy causes elliptical orbits to precess what is good confirmed for Mercury

5.0. Effects of Lorentz transformation

A consequence of the previously adopted (Kyriakos, 2014a) axiomatics of Lorentz-invariant gravitation theory (LIGT) is the assertion that all features of the motion of matter in the gravitational field owed their origin to effects associated with the Lorentz transformations. This means that the elaboration of the equations of gravitation must follow from considering of these effects.

As is well known (Becker, 2013), these questions can be considered without special relativity theory, using only the Maxwell equations.

Effects, that owe their existence to the Lorentz transformations are discussed in many textbooks devoted to the EM theory or SRT (Becker, 2013; Pauli, 1981; et al.). We will not dwell on their withdrawal, and we will only briefly mention some of them.

From the Lorentz transformations follows the velocity transformation, showing that no body can overcome the speed of light. From the Lorentz speed transformations follow the time dilation and length contraction in a moving frame of reference, as well as the transformation of energy and momentum. The use of invariance properties of the wave phase with respect to the Lorentz transformations, allows to obtain the relativistic formula of Doppler effect, aberration, reflection from a moving mirror, Wien's displacement law, etc.

5.1. The transition from Newtonian mechanics to the Lorentz-invariant mechanics

Let us try (Беккер, 2013) to alter the Newtonian equations so that they satisfy the Lorentz transformations. We begin by considering the motion of a particle in a given force field (e.g., electromagnetic or gravitational). Newtonian equations of motion read as follows:

$$m\frac{d\vec{\upsilon}}{dt} = \vec{F}_L, \tag{5.1}$$

where $\vec{F}_L$ is, e.g., the Lorentz force :

$$\vec{F}_L = q\vec{E} - \frac{q}{c}\vec{\upsilon}\times\vec{H}\ , \tag{5.2}$$

Now we will try to give this equation the Lorentz-invariant form. Obviously, the Lorentz-invariant version of the equation (5.1) instead of the classical time t must contain the proper time $\tilde{t}$:

$$m\frac{d\vec{\upsilon}}{d\tau} = \vec{F}_L, \tag{5.1'}$$

In order to find this version of the equation, we replace in (5.1') its proper time in line with the ratio for the Lorentz time dilation $d\tilde{t} = dt\sqrt{1-\beta^2}$ on $dt\sqrt{1-\beta^2}$:

$$m_0\frac{d}{dt}\frac{\vec{\upsilon}}{\sqrt{1-\beta^2}} = q\vec{E} - \frac{q}{c}\vec{\upsilon}\times\vec{H}\ , \tag{5.3}$$

As is known, the equation (5.3) is the Lorentz-invariant equation of motion of a charged particle in an EM field.

Below we will consistently apply this method to obtain the relativistic equations of gravitation in the form of Hamilton-Jacobi equations.

6.0. Solution of the Kepler problem in the framework of LIGT

Two of the most important effects from the point of view of mechanics that arise due to the Lorentz transformations, are the Lorentzian time dilation and contraction of lengths:

$$d\tilde{t} = dt\sqrt{1-\beta^2}\ , \quad d\tilde{r} = \frac{dr}{\sqrt{1-\beta^2}}\ , \tag{6.1}$$

where, as shown in a previous article (Kyriakos, 2014b), $\beta^2 = r_S/r$, and r_S is the Schwarzschild radius.

The free particle motion is described by the Hamilton-Jacobi equation Landau and Lifshitz, 1971):

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - \left(\vec{\nabla}S\right)^2 = m^2c^2\ , \tag{6.2}$$

In a spherical coordinate system (taking into account both relativistic effects) it takes the form:

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial \tilde{t}}\right)^2 - \left(\frac{\partial S}{\partial \tilde{r}}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 - \frac{1}{r^2}\frac{1}{\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2 = m^2c^2\ , \tag{6.3}$$

where $\tilde{t}$ and $\tilde{r}$ are measured in a fixed coordinate system associated with a stationary spherical mass M.

We will start with the account of the first effect

6.1. The equation of motion of a particle in a gravitational field, taking into account the relativistic effect of time dilation

Taking into account that the motion of a particle around the source occurs in the plane, we define this plane by condition $\theta = \pi/2$. In this case, the equation (6.3) takes the form:

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial \tilde{t}}\right)^2 - \left(\frac{\partial S}{\partial \tilde{\vec{r}}}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 = m^2c^2\ , \tag{6.4}$$

Taking into account only the transformation of time $d\tilde{t} = dt\sqrt{1-\beta^2}$ (see (6.1)), equation (6.4) can be rewritten as follows:

$$\frac{1}{1-\beta^2}\left(\frac{\partial S}{\partial t}\right)^2 - c^2\left(\frac{\partial S}{\partial \vec{r}}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 = m^2c^4\ , \tag{6.5}$$

Substituting $1-\beta^2 \approx 1 - r_s/r$, we obtain:

$$\frac{1}{1-\frac{r_s}{r}}\left(\frac{\partial S}{\partial t}\right)^2 - c^2\left(\frac{\partial S}{\partial \vec{r}}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 = m^2c^4\ , \tag{6.6}$$

Let us simplify this equation, taking into account the expansion $1/(1-x) = 1 + x + x^2 + ... + x^n$ for $x << 1$. Since for the actual sizes of the planets and Sun and the distances between them, value $r_s/r << 1$, we can be limited by first two terms of the expansion. At the same time $\frac{1}{1-r_s/r} \cong 1 + r_s/r$, and the equation (6.4) takes the form:

$$\left(1+\frac{r_s}{r}\right)\left(\frac{\partial S}{\partial t}\right)^2 - c^2\left(\frac{\partial S}{\partial \vec{r}}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 = m^2c^4 \ , \qquad (6.7)$$

We will show that L-invariant time dilation leads to the appearance of Newton's gravitational field.

6.1.1 *Newton approach*

Let us present this equation to the nonrelativistic mind, using the transformation $S = S' - mc^2 t$:

$$\left(\frac{\partial S}{\partial t}\right)^2 = \left(\frac{\partial S'}{\partial t}\right)^2 - 2mc^2\frac{\partial S'}{\partial t} + m^2c^4 .$$

Substituting this in (7), we find

$$\left(1+\frac{r_s}{r}\right)\left[\left(\frac{\partial S'}{\partial t}\right)^2 - 2mc^2\frac{\partial S'}{\partial t} + m^2c^4\right] - c^2\left(\frac{\partial S'}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 = m^2c^4 .$$

Expanding the brackets, we obtain:

$$\left(\frac{\partial S'}{\partial t}\right)^2 - 2mc^2\frac{\partial S'}{\partial t} + \frac{r_s}{r}\left(\frac{\partial S'}{\partial t}\right)^2 - 2mc^2\frac{r_s}{r}\frac{\partial S'}{\partial t} + \frac{r_s}{r}m^2c^4 - c^2\left(\frac{\partial S'}{\partial \vec{r}}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 = 0 .$$

Dividing this equation by $2mc^2$, we find:

$$\frac{1}{2mc^2}\left(\frac{\partial S'}{\partial t}\right)^2 - \frac{\partial S'}{\partial t} + \frac{1}{2mc^2}\frac{r_s}{r}\left(\frac{\partial S'}{\partial t}\right)^2 - \frac{r_s}{r}\frac{\partial S'}{\partial t} + \frac{1}{2}\frac{r_s}{r}mc^2 - \frac{1}{2m}\left(\frac{\partial S'}{\partial \vec{r}}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 = 0 , \qquad (6.8)$$

Taking into account that $r_s = \frac{2\gamma M}{c^2}$, we obtain $\frac{1}{2}\frac{r_s}{r}mc^2 = \frac{\gamma mM}{r} = m\varphi_N = -U$, where U is the energy of the gravitational field in the Newtonian theory. In the nonrelativistic case we put $c \to \infty$. Furthermore, for real distances r of the body movement around source with Schwarzschild radius r_s, we have $\frac{r_s}{r} << 1$ and $\frac{r_s}{r}\frac{\partial S'}{\partial t} << \frac{\partial S'}{\partial t}$, and then we can ignore the term $\frac{r_s}{r}\frac{\partial S'}{\partial t}$.

In the limit as $c \to \infty$, equation (6.8) goes over into the classical Hamilton-Jacobi equation for Newton gravitation field

$$\frac{\partial S'}{\partial t} + \frac{1}{2m}\left(\frac{\partial S'}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 = -U , \qquad (6.9)$$

As is known, the solution of this problem leads to a closed elliptical (not preccesing) satellite orbit around the spherical central body.

From this it follows that the inclusion only of Lorentz time dilation into the free Hamilton-Jacobi equation leads to the Kepler problem in non-relativistic theory of gravitation.

Note also that equation (6.9) is a consequence of the L-invariant HJE with the potential field of Newton:

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}+U\right)^2-\left(\frac{\partial S}{\partial r}\right)^2-\frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2-\frac{1}{r^2}\frac{1}{\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2=m^2c^2, \qquad (6.10)$$

Thus, the equations (6.6), (6.9) and (6.10) are equivalent from point of view of their results.

6.2. The equation of motion of a particle in a gravitational field with the Lorentz time dilation and length contraction

Now in order to take into account the length contraction effect along with the effect of time dilation, we will use the Hamilton-Jacobi equation (6.3) in form:

$$\left(\frac{\partial S}{\partial \tilde{t}}\right)^2-c^2\left(\frac{\partial S}{\partial \tilde{r}}\right)^2-\frac{c^2}{r^2}\left[\left(\frac{\partial S}{\partial \theta}\right)^2+\frac{1}{\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2\right]=m^2c^2, \qquad (6.11)$$

Substituting in (6.11) not only $d\tilde{t}=dt\sqrt{1-\beta^2}$, but also $d\tilde{r}=dr\big/\sqrt{1-\beta^2}$, we obtain:

$$\frac{1}{1-\beta^2}\left(\frac{\partial S}{\partial t}\right)^2-c^2\left(1-\beta^2\right)\left(\frac{\partial S}{\partial r}\right)^2-\frac{c^2}{r^2}\left[\left(\frac{\partial S}{\partial \theta}\right)^2+\frac{1}{\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2\right]=m^2c^4, \qquad (6.12)$$

Taking into account that in our theory $1-\beta^2\approx 1-r_s/r$, we obtain from (6.12) the well-known Hamilton-Jacobi equation for general relativity in the case of the Schwarzschild-Droste metric:

$$\frac{1}{1-\frac{r_s}{r}}\left(\frac{\partial S}{\partial t}\right)^2-c^2\left(1-\frac{r_s}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2-\frac{c^2}{r^2}\left[\left(\frac{\partial S}{\partial \theta}\right)^2+\frac{1}{\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2\right]=m^2c^4, \qquad (6.13)$$

As is well known (Landau and Lifshitz, 1971), the solutions of this equation are three well-known effects of general relativity, well confirmed by experiment: the precession of Mercury's orbit, the curvature of the trajectory of a ray of light in the gravitational field of a centrally symmetric source and the gravitational frequency shift of EM waves.

As we noted, in the Kepler problem solution, based on this equation, there is an additional term in the energy, which is missing in Newton's theory:

$$U(r)=-\frac{\gamma_N mM_s}{r}+\frac{M^2}{2mr^2}-\frac{\gamma_N M_s M^2}{c^2mr^3}, \qquad (6.14)$$

which is responsible for the precession of the orbit of a body, rotating around a spherically symmetric stationary center. From the above analysis it follows that the appearance of this term is provided by Lorentz effect of the length contraction.

We found above that the term $\frac{1}{1-r_s/r}\left(\frac{\partial S'}{\partial t}\right)^2$ containing the Lorentz time dilation effect in the classical approximation leads to the equation of Newton gravitation with Newton's gravitational energy. From this it follows that the precession of the orbit ensure the introduction of an additional term $c^2\left(1-\frac{r_s}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2$.

6.3. Gravitational deflection of light ray trajectory

The path of a light ray (Landau and Lifshitz, 1971, p. 308-309) in a centrally symmetric gravitational field is determined by the eikonal equation

$$g^{ik}\frac{\partial\Psi}{\partial x^i}\frac{\partial\Psi}{\partial x^k}=0, \qquad (6.15)$$

which differ from Hamilton-Jacobi equation only in having $m=0$, at the same time, in place of the energy $\varepsilon_p=-\partial S/\partial t$ of the particle we must write the frequency of the light $\omega_\lambda=-\partial\Psi/\partial t$.
The solution show that under the influence of the field of attraction the light ray is bent: its trajectory is a curve, which is concave toward the center (the ray is 'attracted' toward the center), so that the angle between its two asymptotes differs from π by

$$\delta\varphi=\frac{2r_s}{\rho}=\frac{4\gamma_N M_S}{c^2\rho}, \qquad (6.16)$$

In other words, the ray of light, passing at a distance ρ from the center of the field, is deflected through an angle $\delta\varphi$.

6.4. Gravitational time dilation and frequency red shift

We are able to prove a general statement regarding the influence of a gravitational field on clocks (Pauli, 1981). Let us take a reference system *K* which rotates relative to the Galilean system K_0 with angular velocity ω. A clock at rest in *K* will then be slowed down the more, the farther away from the axis of rotation the clock is situated, because of the transverse Doppler effect. This can be seen immediately by considering the process as observed in system K_0. The time dilatation is given by

$$t=\frac{\tau}{\sqrt{1-\frac{\upsilon^2}{c^2}}}=\frac{\tau}{\sqrt{1-\frac{\omega^2 r^2}{c^2}}}, \qquad (6.17)$$

The observer rotating with *K* will not interpret this shortening of the time as a transverse Doppler effect, since after all the clock *is* at rest relative to him. But in *K* a gravitational field (field of the centrifugal force) exists with potential $\varphi=-\frac{1}{2}\omega^2 r^2$.

Thus the observer hi *K* will come to the conclusion that the clocks will be slowed down the more, the smaller the gravitational potential at the particular spot. In particular, taking into account that $\upsilon^2/c^2=\beta^2=r_s/r=2\varphi/c^2$, the time dilatation Δt is given, to a first approximation, by

$$t=\frac{\tau}{\sqrt{1+\frac{2\varphi}{c^2}}}\sim\tau\left(1-\frac{\varphi}{c^2}\right);\quad \frac{\Delta t}{\tau}=-\frac{\varphi}{c^2}, \qquad (6.18)$$

Einstein[293] applied an analogous argument to the case of uniformly accelerated system. We thus see that the transverse Doppler effect and the time dilatation produced by gravitation appear as two different modes of expressing the same fact, namely that a clock will always indicate the proper time $\tau = \frac{1}{ic}\int ds$.

Relation (6.18) has an important consequence which can be checked by experiment. The transport of clocks can also be effected by means of a light ray, if one regards the vibration process of light as a clock.

If, therefore, a spectral line produced in the sun is observed on the earth, its frequency will, according to (6.18), be shifted towards the red compared with the corresponding terrestrial frequency. The amount of this shift will be

$$\frac{\Delta \nu}{\nu} = -\frac{\varphi_E - \varphi_S}{c^2}, \tag{6.19}$$

where φ_E is the value of the gravitational potential on the earth, φ_S that on the surface of the sun. The numerical calculation gives $\frac{\Delta \nu}{\nu} = 2{,}12 \cdot 10^{-6}$, corresponding to a Doppler effect of 0,63 km/sec.

Einstein (Einstein, 1911) applied an analogous argument to the case of uniformly accelerated system.

Let us assume (Sivukhin, 2005) that the clock *A* relatively to the system *S* is moving with constant acceleration a. We will count the time t from the moment when the velocity was zero. Then $\upsilon = \sqrt{2ax}$, where *x* is the distance that the clock A covered during the time t. Therefore:

$$dt = dt_0 \Big/ \sqrt{1 - 2ax/c^2}, \tag{6.20}$$

Now let us introduce an accelerated reference frame S_0, which moves together with the clock *A*. In this system the clock *A* is immobile, but there are inertial forces. If all the phenomena will be described, taking S_0 as a reference frame, then as the cause of time dilation t_0 the inertial forces should be considered. The inertial force per unit mass of the moving body is $-a$. But, according to the principle of equivalence, the inertial forces are indistinguishable from the gravitational field, the intensity of which in our case is $\vec{g} = -\vec{a}$. Then the gravitational potential is $\varphi = -gx$ and the formula (6.20) becomes:

$$dt = dt_0 \Big/ \sqrt{1 - 2\varphi/c^2} \approx dt_0 \left(1 - \varphi/c^2\right), \tag{6.21}$$

or

$$\frac{dt - dt_0}{dt_0} = -\frac{\varphi}{c^2}, \tag{6.22}$$

As zero gravitational potential, the potential of point is considered, at which the moving and stationary clocks run equally fast. Therefore, in formulas (6.21) and (6.22), the time interval dt can be counted not by the clock of the inertial system *S*, but by the clock that is in rest in system S_0, which is located at the point *B* with zero potential. In general, we can set the initiation of count of gravitational potential at any point, if the formula (6.22) has the form:

$$\frac{dt_{0A} - dt_{0B}}{dt_{0A}} = -\frac{\varphi_B - \varphi_A}{c^2}, \qquad (6.23)$$

where the time intervals dt_{0A} and dt_{0B} are counted by two clocks, which are in rest in an accelerated reference frame S_0 at points *A* and *B* with gravitational potentials φ_A and φ_B.

Conclusion

Thus, we can say that, in the case of centrally symmetric gravitational field, within the framework of LIGT we get the same results as in the framework of general relativity. It is noteworthy that in order to obtain these results minor adjustments in Newton's theory are required, which are ensured by two effects following from the Lorentz transformations.

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