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Spherically Symmetric String Cosmological Model with Magnetic Field Admitting Conformal Motion

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Abstract

In this paper we have investigated spherically symmetric cosmological model for string cloud and string fluid in the presence of electromagnetic field in the context of general relativity. The solution of the Einstein field equations have been obtained under the assumption of the one parametric group of conformal motion.

Keywords: String cosmological model, Spherical symmetric space time, Magnetic field, conformal motion.

1. Introduction

At the very early stages of evolution of the universe, it is generally assumed that during the phase transition the symmetry of the universe is broken spontaneously. It can give rise to topological stable defects such as domain walls, strings and monopoles. Of all these cosmological structures, cosmic strings have excited the most interest. Cosmic strings arise during phase transitions after the big bang explosion as the temperature goes down below some critical temperature [Zel'dovich(1975) , Kibble(1976), Vilenkin (1981)].

The string theory developed to describe an event at the early stage of evolution of the universe. These strings have stress energy and couple in a simple way to the gravitation field and hence it is interesting to study the gravitational effect that arises from strings. In fact, the general relativistic treatment was initiated by Letelier (1983) and Satchel(1980). Letelier(1979) has obtained the solution to the Einstein field equations for a cloud of string with spherical, plane, cylindrical symmetry and obtained cosmological models in Bianchi I and Kantowski-Sachs space time . In the strings theory, the myriad of particle types is replaced by a single fundamental

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building block, a 'string'. These strings can be closed, like loops, like a hair. As the string moves through times it traces out a sheet, according to whether it is closed or open. Furthermore, the string is free to vibrate and different vibrational nodes of the string represent the different particle types, since different nodes are seen as different masses or spins. One mode of vibration, or note, makes the string appear as an electron, another as a photon there is even a mode describing the graviton, the particle carrying the force of gravity, which is an important reason why string theory has received so much attention. The point is that we can make sense of the interaction of two gravitons in string theory in a way we could not in QFT. There are no infinites. And gravity is not something we put in by hand. It has to be there in a theory of strings. So, the first great achievement of string theory was to give a consistent theory of gravity, which resembles GR at microscopic distances. Moreover string theory possess the necessary degree of freedom to describe the other interactions. At this point a great hope was created that string theory would be able to unify all the known forces and particles together in to a single 'Theory of Everything's'.

The magnetic field has important role at the cosmological scale and is present in galactic and intergalactic spaces. The importance of the magnetic field for various astrophysical phenomenon's has been studied in many papers. Melvin (1975) has pointed out that during the evolution of the universe, the matter is in highly ionized state and is smoothly coupled with the field, subsequently forming neutral matter as a result of universe expansion. Therefore considering the presence of magnetic field in strings universe is not unrealistic and has been investigated by many authors [Banerjee (1990), Shri Ram (1995), Singh (1999), Bali (2003)]. Herrera and Leon (1985) have obtained exact analytical solution of the static Einstein-Maxwell equation for perfect and anisotropic fluid under the assumption of spherically symmetry and the existence of the one-parameter group of conformal motion.

Herrera et al. (1984) studied the consequences of the existence of a one parameter group of conformal motion for anisotropic matter in the context of Einstein's general relativity and obtained analytical solutions of field equations for static and spherically symmetric distribution of isotropic and anisotropic matter. Yavuz and Yilmaz (1997) have solved the Einstein field equations via conformal motions spherically symmetric space- times in the context of string.

Yavuz et al. (2005) have examined charged strange quark matter attached to the string cloud in the spherical symmetric space-time admitting one parameter group of conformal motions.
In this paper we have obtained solution of gravitational field equations for static spherically space-times with cloud and fluid string source one parameter group of conformal motions in the presence of magnetic field.

The energy- momentum tensor for a cloud of string with electromagnetic field can be written as

$$T_{ij} = \rho u_i u_j - \lambda x_i x_j + E_{ij} \tag{1}$$

where ρ is the rest energy for the cloud of strings with particles attached to them and λ is string tensor density, they are related by

$$\rho = \rho_p + \lambda \tag{2}$$

Here ρ_p is the particle energy density. The unit time like vector u^i describes the cloud four velocities and the unit space vector x^i represents the direction of anisotropy, i.e. the string's direction [Letelier (1983)].
The energy momentum tensor for a fluid of string [Letelier (1980), (1981)] with electromagnetic field is

$$T_{ij} = (q + \rho_s)(u_i u_j - x_i x_j) + q g_{ij} + E_{ij} \tag{3}$$

Also, note that

$$u^i u_i = -x^i x_i = -1 \text{ and } u^i x_i = 0 \tag{4}$$

where ρ_s is string density and q is "string tension " and also "pressure",
E_{ij} is the energy momentum for magnetic field given by

$$E_{ij} = \frac{1}{4\pi}\left(F_{i\alpha} F_{j\beta} g^{\alpha\beta} - \frac{1}{4} g_{ij} F^{\alpha\beta} F_{\alpha\beta} \right) \tag{5}$$

where F_{ij} is the electromagnetic field tensor which satisfied the Maxwell equations

$$F_{[ij;\alpha]} = 0, \ \left(F^{ij}\sqrt{-g}\right)_{;j} = 0 \tag{6}$$

In commoving coordinates, the incident magnetic field is taking along x-axis, with the help of Maxwell equations (6), the only non- vanishing component of F_{ij} is

$$F_{23} = const = H \tag{7}$$

The metric for static spherically symmetric space time can be written as

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) \tag{8}$$

with the convention $x^1 = r, x^2 = \theta, x^3 = \phi, x^4 = t$ and where A and B are only functions of r.

2. Solutions of string cosmology with magnetic field in static spherical symmetric

2.1. Field equations for a cloud of strings with magnet

Einstein's field equations for (1) are

$$R_{ij} - \frac{1}{2} R g_{ij} = -\left(\rho u_i u_j - \lambda x_i x_j + E_{ij}\right) \tag{9}$$

We use geometries units so that $8\pi G = c = 1$. In a commoving coordinate system we may choose

$$u^i = (o, o, o, \frac{1}{A}); x^i = (\frac{1}{B}, 0, 0, 0) \tag{10}$$

The field equations (9) for the metric (8) lead to the following system of equations

$$\frac{1}{B^2}\left(\frac{2A'}{Ar} + \frac{1}{r^2}\right) - \frac{1}{r^2} = -\lambda - \frac{H^2}{r^4} \tag{11}$$

$$\frac{1}{B^2}\left[\frac{A''}{A} - \frac{A'B'}{AB} + \frac{1}{r}\left(\frac{A'}{A} - \frac{B'}{B}\right)\right] = \frac{H^2}{r^4} \tag{12}$$

$$\frac{1}{B^2}\left(\frac{2B'}{Br} - \frac{1}{r^2}\right) + \frac{1}{r^2} = \rho + \frac{H^2}{r^4} \tag{13}$$

where a prime denoted differentiation with respect to r.

Due to the non linearity of the field equations (11) to (13), it is rather difficult to obtain a solution in its generality and therefore one has to make a certain simplifying assumption to derive useful results. The assumptions are motivated either physical consideration or by mathematical convenience. In the next section, we assume that the physical metric (8) admit one parameter group of conformal motions and obtain the solution of the field equations.

2.1.1. The conformal motions and solution of the field equations

A space time (8) admit one parameter group of conformal motions generated by the vector field ξ^i if

$$L_\xi g_{ij} = 2\psi g_{ij}, \psi = \psi(x^i) \tag{14}$$

where L_ξ signifies the Lie derivative along ξ^i and $\psi(x^i)$ is the conformal factor.

By virtue of the spherical symmetry and independence of the metric tensor on the time coordinate the most general form ξ^i is

$$\xi^i = lx^i \tag{15}$$

where l is an arbitrary function of r.

The functions A and B are restricted by the condition (14). From equations (8), (14, (15), we have

$$\frac{B^{'}}{B}\xi^1 + \xi^{''} = \psi \tag{16}$$

$$\xi^1 = \psi . r \tag{17}$$

$$\frac{A^{'}}{A}\xi^1 = \psi \tag{18}$$

where $\xi^2 = \xi^3 = \xi^4 = 0$.

As a consequence from equations (15) to(18) we get

$$A = c_1 r \tag{19}$$

$$B = \frac{c_2}{\psi}, c_2 > 0 \tag{20}$$

$$l = c_2 r \tag{21}$$

where c_1 and c_2 are non –zero integration constants.

Feeding (19) to (21) back into the field equations (11) to (13) we get,

$$\lambda = \frac{1}{r^2}\left(1 - \frac{H^2}{r^4} - \frac{3\psi^2}{c_2{}^2}\right) \tag{22}$$

$$2\psi\psi' + \frac{\psi^2}{r} - \frac{H^2 c_2^{\,2}}{r^3} = 0 \tag{23}$$

$$\rho = \frac{1}{r^2}\left(1 - \frac{H^2}{r^2}\right) - \frac{\psi^2}{c_2^{\,2}}\left(\frac{2\psi'}{\psi r} + \frac{1}{r^2}\right) \tag{24}$$

From (23) we have

$$\psi = \sqrt{\frac{c_3^{\,2}}{r} - \left(\frac{Hc_2}{r}\right)^2} \tag{25}$$

where c_3 is an integration constant. Substituting equation (25) into (20), (22), and (24) we have , respectively,

$$B = \frac{c_2}{\sqrt{\frac{c_3^{\,2}}{r} - \left(\frac{Hc_2}{r}\right)^2}} \tag{26}$$

$$\lambda = \frac{1}{r^2}\left[1 - \frac{3}{r}\left(\frac{c_3}{c_2}\right)^2 + 2\left(\frac{H}{r}\right)^2\right] \tag{27}$$

$$\rho = \frac{1}{r^2}\left[1 - 2\left(\frac{H}{r}\right)^2\right] \tag{29}$$

So, we have the line element in the form

$$ds^2 = -(c_1 r)^2 dt^2 + \left[\frac{c_2^{\,2}}{c_3^{\,2} - \left(\frac{Hc_2}{r}\right)^2}\right] r dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

(30)

These solutions are different from the solutions given by Letelier (1979) in the absence of magnetic field. Here we have solved Einstein's field equations via CKV for the cloud of strings with particles attached to them in the presence of magnetic field.

2.2. Field equations for fluids of strings

The Einstein equation for (3) is

$$R_{ij} - \frac{1}{2} R g_{ij} = -\left[(q + \rho_s)(u_i u_j - x_i x_j) + q g_{ij} + E_{ij} \right] \tag{31}$$

Again we choose

$$u^i = (o, o, o, \frac{1}{A}) \qquad x^i = (\frac{1}{B}, 0, 0, 0)$$

The field equations (31) for the metric (8) lead to the following system of equations

$$\frac{1}{B^2}\left(\frac{2A'}{Ar} + \frac{1}{r^2} \right) - \frac{1}{r^2} = -\rho_s - \frac{H^2}{r^4} \tag{32}$$

$$\frac{1}{B^2}\left[\frac{A'B'}{AB} - \frac{A''}{A} - \frac{1}{r}\left(\frac{A'}{A} - \frac{B'}{B} \right) \right] = -q - \frac{H^2}{r^4} \tag{33}$$

$$\frac{1}{B^2}\left(\frac{-2B'}{Br} + \frac{1}{r^2} \right) - \frac{1}{r^2} = -\rho_s - \frac{H^2}{r^4} \tag{34}$$

where a prime denotes differentiation with respect to r.

2.2.1. The conformal motions and solution of the field equations

From equation (32) and (34) we have

$$A = \frac{c_4}{B}, \tag{35}$$

where c_4 is nonzero integration constant. As the consequence of equations (14) to (18), we get

$$A = c_1 r \tag{36}$$

$$B = \frac{c_2}{\psi}, \; c_2 > 0 \tag{37}$$

$$l = c_2 r \tag{38}$$

From equations (35) and (36) , we have

$$B = \frac{c_4}{c_1}\left(\frac{1}{r} \right) \tag{39}$$

Next, from equation (37) and (39), we found

$$\psi = \frac{c_1 c_2}{c_4} r \tag{40}$$

Substituting equations (36) and (39) into (32) to (34), we have respectively,

$$\rho_s = \frac{1}{r^2}\left(1-\left(\frac{H}{r}\right)^2\right)-3\left(\frac{c_1}{c_4}\right)^2 \tag{41}$$

$$q = 3\left(\frac{c_1}{c_4}\right)^2 - \frac{H^2}{r^4} \tag{42}$$

In this case, we have the line element in the form

$$ds^2 = -(c_1 r)^2 dt^2 + \left(\frac{c_4}{c_1}\frac{1}{r}\right)^2 dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{43}$$

These solutions are also different from the solutions given by Letelier (1981) in the absence of magnetic field.

Here we have solved the Einstein's field equations for the fluids of string and obtained ρ_s and q exactly in the presence of magnetic field.

If the function ψ and an equation of state for the stresses are specified a priori, the problem will be fully determined. So, We will examine the following physically meaningful four cases depending on $\psi(r)$.

Case i) if $\psi = c_3 r$ then from equations (22)-(24) we get

$$\rho = \lambda = \frac{1}{r^2} - \left(\frac{c_3}{c_2}\right)^2 \frac{1}{r^3} - 2\left(\frac{c_3}{c_2}\right)^2 \tag{44}$$

$$H^2 = \left(\frac{c_3}{c_2}\right)^2 r - \left(\frac{c_3}{c_2}\right)^2 r^4 \tag{45}$$

$$\rho_p = 0 \tag{46}$$

where c_3 is integrating constant.

Case ii) If $\psi = \frac{c_3}{\sqrt{r}}$,from equations (22)- (24) we get following expression

$$\rho = \frac{1}{r^2} \tag{47}$$

$$\lambda = \frac{1}{r^2} - 3\left(\frac{c_3}{c_2}\right)^2 \frac{1}{r^3} \tag{48}$$

$$H^2 = 0 \tag{49}$$

$$\rho_p = 3\left(\frac{c_3}{c_2}\right)^2 \frac{1}{r^3} \tag{50}$$

Case iii) If $\psi = \sqrt{\frac{1}{2}\left(c_2{}^2 - \frac{c_3{}^2}{r}\right)}$, from equations (22)-(24) we get following expressions

$$\rho = \rho_p = \frac{1}{r^2} - 2\left(\frac{c_3}{c_2}\right)^2 \frac{1}{r^3} \tag{51}$$

$$H^2 = \frac{3}{2}\left(\frac{c_3}{c_2}\right)^2 r - \frac{r^2}{2} \tag{52}$$

$$\lambda = 0 \tag{53}$$

Case iv) If $\psi = \sqrt{c_2{}^2\left(\log r + \frac{1}{2r}\left(\frac{c_3}{c_2}\right)^2\right)}$, from equations (22)-(24) we get following expressions.

$$\lambda = \frac{1}{r^2}(1 - 2\log r) - 2\left(\frac{c_3}{c_2}\right)^2 \frac{1}{r^3} \tag{54}$$

$$\rho = 0 \tag{55}$$

$$\rho_p = -\lambda = -[\frac{1}{r^2}(1 - 2\log r) - 2\left(\frac{c_3}{c_2}\right)^2 \frac{1}{r^3}] \tag{56}$$

$$H^2 = \frac{1}{2}\left(\frac{c_3}{c_2}\right)^2 r - \log r \tag{57}$$

3. Conclusion

In the case of string cloud, the Einstein field equation has been solved by spherically symmetric metric space time with string with the assumption that a space time admits one parameter group of conformal motion. At the limit case $r \to 0$ and $r \to \infty, \rho, \rho_p$ and λ are infinite and zero respectively i.e. the solutions are reasonable physically. ρ and ρ_p are positive throughout the evolution of the universe i.e. energy conditions are satisfied. For $c_3 \to 0$ and in the absence of magnet, we have a geometric string $\rho \to \lambda$. When $r \to 0$, we have $\frac{\rho_p}{|\lambda|} \to 2$. In this case the distribution of particle is twice the strings. Also, $\frac{\rho_p}{|\lambda|} \to 0$ as $r \to \infty$. In this case ρ_p decays more rapidly than λ.

In the case of string fluid q is positive if $3\left(\frac{c_1}{c_4}\right)^2 > \frac{H^2}{r^4}$ and constant in the absence of magnetic field. Also, we observe that $\rho_s + q \to \rho$ when as $c_1 \to 0$ and in the absence of magnets $\rho_s \to \lambda$ (i.e. geometric string) when as $c_3 \to 0$.

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