**Article**

# [z-t]-Type Plane Wave Solutions of Weakened Field Equations

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## Abstract

In this paper we have proved that the purely plane gravitational wave $g_{ij}$ be the solutions of the Weakened Field Equations (WFE), in general relativity.

**Keywords:** WEF, plane gravitational waves, curvature tensor, Ricci tensor, Weyl curvature tensor.

## 1. INTRODUCTION

The plane gravitational waves $g_{ij}$ are mathematically exposed by H.Takeno [1], in general relativity. S.N.Pandey [3] has proved that, the space-time,

$$ds^2 = -A\,dx^2 - 2\,D\,dx\,dy - B\,dy^2 - dz^2 + dt^2, \tag{1.1}$$

where A,B,D are the functions of Z = (z – t), be the solutions of the five WFE (I) – (V).

$$I_{ijk} = R^a{}_{ijk;a} = 0, \tag{I}$$

$$(-g)^{1/4}\,[g^{ih} R_{kj;ih} - g^{ih} R_{ij;kh} + (1/6) R_{;kj} - (1/6) g_{jk} g^{ih} R_{;ih} - R^{ih} C_{jhik}$$
$$+ (R/6) g^{ih} C_{jhik}] = 0, \tag{II}$$

$$(-g)^{1/2}\,[g^{hj} g^{ki} \{ 2R_{jlim} R^{ml} + g^{ml} R_{ij;lm} - R_{;ij}\} - (1/2) g^{hk} ( R^l{}_m R^m{}_l - g^{lm} R_{;lm})] = 0, \tag{III}$$

$$(-g)^{1/2}\,[ (g^{hk} g^{tu} - (1/2) g^{ht} g^{ku} - (1/2) g^{hu} g^{kt}) R_{;ut} + R( R^{kh} - (1/4) g^{kh} R )] = 0, \tag{IV}$$

$$\Theta^{ij}{}_k = R^{ij}{}_{;k} = 0, \tag{V}$$

where $C_{jhik}$ is Weyl curvature tensor & semicolon (;) denotes the covariant derivative. These field equations are solved by Lovelock [2] & they are originally suggested by Kilmister and Newman, Pirani, Rund, Eddington & Rund respectively.

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In this paper we have proved that the plane waves $g_{ij}$ given by the space-time

$$ds^2 = -A\,dx^2 - 2D\,dx\,dy - B\,dy^2 - (C-E)\,dz^2 - 2E\,dz\,dt + (C+E)\,dt^2, \qquad (1.2)$$

where A,B,C,D,E are the functions of $Z = (z - t)$ satisfying $A,B > 0$, $C > |E|$, be the solutions of the WFE (I) – (V).

## 2. DEFINITION

The plane gravitational waves $g_{ij}$ are defined as the non-flat solutions of the field equation

$$R_{ij} = 0; \qquad i,j = 1,--,4, \qquad (2.1)$$

in an empty region of the space-time with

$$g_{ij} = g_{ij}(Z); \quad Z = Z(x, y, z, t), \qquad (2.2)$$

in some suitable coordinates system such that

$$g^{ij} Z_{,i} Z_{,j} = 0; \qquad Z_{,i} = \frac{\partial Z}{\partial x^i} \qquad (2.3)$$

such that $Z_{,i} \neq 0$.

The signature convention adopted is as follows,

$$g_{ll} < 0; \qquad \begin{vmatrix} g_{ll} & g_{lk} \\ g_{kl} & g_{kk} \end{vmatrix} > 0; \qquad \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} < 0; \qquad g_{44} > 0. \qquad (2.4)$$

(No summation for l,k = 1,2,3).

And accordingly $g = \det(g_{ij}) < 0$. (2.5)

## 3. SOLUTIONS OF THE WFE

From (1.2), we have,

$$g^{ij} = \begin{bmatrix} -\frac{B}{m} & \frac{D}{m} & 0 & 0 \\ \frac{D}{m} & -\frac{A}{m} & 0 & 0 \\ 0 & 0 & \frac{-C-E}{C^2} & -\frac{E}{C^2} \\ 0 & 0 & -\frac{E}{C^2} & \frac{C-E}{C^2} \end{bmatrix} \tag{3.1}$$

where m = $AB - D^2 > 0$.

From (1.2) & (3.1), the non-vanishing components of the Christoffel's symbols are as follow,

$$\left.\begin{aligned}
&\Gamma^1_{13} = -\Gamma^1_{14} = \frac{1}{2m}(B\bar{A} - D\bar{D}), \qquad \Gamma^1_{23} = -\Gamma^1_{24} = \frac{1}{2m}(B\bar{D} - D\bar{B}), \\
&\Gamma^2_{13} = -\Gamma^2_{14} = \frac{1}{2m}(A\bar{D} - D\bar{A}), \qquad \Gamma^2_{23} = -\Gamma^2_{24} = \frac{1}{2m}(A\bar{B} - D\bar{D}), \\
&\Gamma^3_{11} = \Gamma^4_{11} = -\frac{\bar{A}}{2C}, \qquad \Gamma^3_{12} = \Gamma^4_{12} = -\frac{\bar{D}}{2C}, \\
&\Gamma^3_{22} = \Gamma^4_{22} = -\frac{\bar{B}}{2C}, \\
&\Gamma^3_{33} = -\Gamma^3_{34} = \Gamma^3_{44} = \frac{1}{2C^2}[2E\bar{C} + C(\bar{C} - \bar{E})], \\
&\Gamma^4_{33} = -\Gamma^4_{34} = \Gamma^4_{44} = \frac{1}{2C^2}[2E\bar{C} - C(\bar{C} + \bar{E})].
\end{aligned}\right\} \tag{3.2}$$

Using (1.2), (3.1) and (3.2), the non-vanishing components of the curvature tensor

$R_{ijkl}$ and Ricci tensor $R_{ij}$ are obtained as follow,

$$\left.\begin{aligned}
&R_{1313} = -R_{1314} = R_{1414} = \frac{\bar{\bar{A}}}{2} - \frac{1}{4m}[B\bar{A}^2 + A\bar{D}^2 - 2D\bar{A}\,\bar{D}] - \frac{\bar{A}\,\bar{C}}{2C} = u, \\
&R_{1323} = -R_{1324} = -R_{1423} = R_{1424} = \frac{\bar{\bar{D}}}{2} - \frac{1}{4m}[B\bar{A}\,\bar{D} + A\bar{B}\,\bar{D} - D\bar{A}\,\bar{B} - D\bar{D}^2] - \frac{\bar{C}\,\bar{D}}{2C} \\
&\qquad = w, \\
&R_{2323} = -R_{2324} = R_{2424} = \frac{\bar{\bar{B}}}{2} - \frac{1}{4m}[A\bar{B}^2 + B\bar{D}^2 - 2D\bar{B}\,\bar{D}] - \frac{\bar{C}\,\bar{B}}{2C} = v, \text{ and}
\end{aligned}\right\} \tag{3.3}$$

$$\left.\begin{aligned}
&R_{33} = -R_{34} = R_{44} = P = \frac{1}{m}(A\,v + B\,u - 2\,D\,w). \\
&\text{Also, } R^{33} = R^{34} = R^{44} = \frac{P}{C^2}.
\end{aligned}\right\} \tag{3.4}$$

By using (3.1), (3.3) and (3.4), we deduced,

a) $R = 0$, b) $g = -C^2(AB - D^2)$, c) $R^l_m R^m_l = 0$, d) $R_{jlim} R^{ml} = 0$. (3.5)

$$\left.\begin{aligned} &\text{Also, by (3.4), } R_{33;11} = R_{33;12} = R_{33;22} = R_{34;11} = R_{34;12} = R_{34;22} = R_{44;11} = \\ &\qquad = R_{44;12} = R_{44;22} = 0, \\ &\text{and } R_{33;33} = -R_{33;34} = R_{33;44} = -R_{34;33} = R_{34;34} = -R_{34;44} = \\ &\qquad = R_{44;33} = -R_{44;34} = R_{44;44} = Q = \bar{\bar{P}} - \frac{2\bar{P}\bar{\bar{C}}}{C} - \frac{5\bar{P}\bar{C}}{C} + \frac{8P\bar{C}^2}{C^2}. \end{aligned}\right\} \quad (3.6)$$

Now we shall prove the gravitational plane waves $g_{ij}$ given by (1.2) be the solutions of the WFE ( I ) – (V) in the form of theorems as follow.

**Theorem 1:** Prove that the plane wave $g_{ij}$ given by (1.2) be the solutions of WFE

(I), (II) and (IV).

*Proof:* The curvature tensor $R^a_{ijk}$ satisfies the Bianchi identity

$$R^a_{ijk;m} + R^a_{ikm;j} + R^a_{imj;k} = 0. \quad (3.7)$$

Contracting a with m, we get,

$$R^a_{ijk;a} + R_{ik;j} - R_{ij;k} = 0. \quad (3.8)$$

But from (3.4), we get, $R_{ik;j} - R_{ij;k} = 0$, (3.9)

hence from (3.8), we get, $R^a_{ijk;a} = 0$.

So, WFE (I) is satisfied.

Also, from (3.9), it follows that,

$$R_{ik;jh} = R_{ij;kh}\ , \quad (3.10)$$

using (3.10) & (3.5), WFE (II) reduces to,

$$(-g)^{1/4}\,[g^{ih} R_{kj;ih} - g^{ih} R_{ij;kh} - R^{ih} C_{jhik}] = 0, \quad (3.11)$$

which on simplification, becomes

$$(-g)^{1/4}\ R^{ih} C_{jhik} = 0, \quad (3.12)$$

by the virtue of (3.5), (3.12) is identically satisfied.

Also, WFE (IV) is satisfied by a) in (3.5). Hence the theorem.

**Theorem 2:** A necessary and sufficient condition that $g_{ij}$ given by (1.2) be a solutions of WFE (III) is Q = 0,

where $Q = \bar{\bar{P}} - \frac{2P\bar{\bar{C}}}{C} - \frac{5\bar{P}\bar{C}}{C} + \frac{8P\bar{C}^2}{C^2}$.

[ Bar (–) over a letter denotes the derivative with respect to Z.]

*Proof:* Let $g_{ij}$ given by (1.2) be the solutions of WFE (III).

By the virtue of (3.5), (III) reduces to

$$(-g)^{1/2} g^{hj} g^{ki} g^{ml} R_{ij;lm} = 0, \tag{3.13}$$

(3.13) is identically satisfied for all values of h,k expect for h,k = 3,4.

When h,k = 3,4, equation (3.13), on simplication gives,

$$Q = \bar{\bar{P}} - \frac{2P\bar{\bar{C}}}{C} - \frac{5\bar{P}\bar{C}}{C} + \frac{8P\bar{C}^2}{C^2} = 0, \quad \text{by (3.4) \& (3.6).}$$

Conversely, if $Q = \bar{\bar{P}} - \frac{2P\bar{\bar{C}}}{C} - \frac{5\bar{P}\bar{C}}{C} + \frac{8P\bar{C}^2}{C^2} = 0.$

$$\Rightarrow (-g)^{1/2} g^{hj} g^{ki} g^{ml} R_{ij;lm} = 0, \quad \text{by (3.4) \& (3.6).} \tag{3.14}$$

LHS of (III) = 0, by (3.5) and (3.14).

So, WFE (III) is identically satisfied. Hence the theorem.

**Theorem 3:** A necessary and sufficient condition that $g_{ij}$ given by (1.2) be the solutions of WFE (V), is $[\overline{P/C^2}] = 0$.

*Proof:* Let $g_{ij}$ given by (1.2) be the solutions of WFE (V).

Equation (V), implies $R^{ij}{}_{;k} = 0$

$$\Rightarrow \frac{\partial R^{ij}}{\partial x^k} + \Gamma^i_{sk} R^{sj} + \Gamma^j_{sk} R^{is} = 0. \tag{3.15}$$

Equation (3.15) is identically satisfied for all values of i, j, k, except for i,j,k = 3,4,

by using the components of $R^{ij}$ & Christoffel's symbols $\Gamma^{i}_{sk}$.

Equation (3.15) for i, j, k = 3,4 gives

$$[\overline{P/C^2}] = 0, \text{ by (3.4).} \tag{3.16}$$

Conversely, if $[\overline{P/C^2}] = 0$,

$$\Rightarrow \frac{d}{dZ}[P/C^2] = 0$$

$$\Rightarrow \frac{d}{dZ}[P/C^2]\frac{\partial Z}{\partial x^k} = 0, \text{ for all k= 1,2,3,4.}$$

$$\Rightarrow \frac{\partial}{\partial x^k}[P/C^2] = 0.$$

$$\Rightarrow [P/C^2],_k = 0. \tag{3.17}$$

Hence,

$$R^{ij}{}_{;k} = \frac{\partial R^{ij}}{\partial x^k} + \Gamma^i_{sk}R^{sj} + \Gamma^j_{sk}R^{is}$$

$$= [P/C^2],_k + \Gamma^i_{sk}R^{sj} + \Gamma^j_{sk}R^{is} \text{ by (3.4)},$$

$$= 0, \text{ by (3.2),(3.4) \& (3.17).}$$

$$\Rightarrow R^{ij}{}_{;k} = 0.$$

Implies $g_{ij}$ be the solutions of WFE (V). This proves the theorem.

## CONCLUSION

The plane waves $g_{ij}$ given by (1.2) be the solutions of the WFE (I) – (V). Also we note that, [3] is the special case (when E = 0, C = 1) of this paper.

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