Exploration

Birational Maps as Morphisms of Cognitive Structures

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Abstract

Birational maps and their inverses are defined in terms of rational functions. They are very special in the sense that they map algebraic numbers in a given extension E of rationals to E itself. In the TGD framework, E defines a unique discretization of the space-time surface if the preferred coordinates of the allowed points belong to E. I refer to this discretization as cognitive representation. Birational maps map points in E to points in E so that they define what might be called cognitive morphism. M^8-H duality duality ($H=M^4\times CP_2$) relates the number vision of TGD to the geometric vision. M^8-H duality maps the 4-surfaces in M_c^8 to space-time surfaces in H: a natural condition is that in some sense it maps E to E and cognitive representations to cognitive representations. There are special surfaces in M_c^8 that allow cognitive explosion in the number-theotically preferred coordinates. M^4 and hyperbolic spaces H^3 (mass shells), which contain 3-surfaces defining holographic data, are examples of these surfaces. Also the 3-D light-like partonic orbits defining holographic data. Possibly also string world sheets define holographic data. Does cognitive explosion happen also in these cases? In M_c^8 octonionic structure allows to identify natural preferred coordinates. In H, in particular M^4, the preferred coordinates are not so unique but should be related by birational mappings. So called Hamilton-Jacobi structures define candidates for preferred coordinates: could different Hamilton-Jacobi structures relate to the each other by birational maps? In this article these questions are discussed.

1 Introduction

Birational maps (see this) and their inverses are defined in terms of rational functions. They are very special in the sense that they map algebraic numbers in a given extension E of rationals to E itself.

1. In the TGD framework, the algebraic extensions E are defined by rational polynomials P at the level of M_c^8 identifiable as complexified octonions. E defines a unique discretization for the number theoretically preferred coordinates of M_c^8 by the condition that the M^8 coordinates have values in E: I call these discretizations cognitive representations. They make sense also in the extensions of p-adic number fields induced by E serving as correlates of cognition in TGD inspired theory of conscious experience. Birational maps respect the extension E associated with the cognitive representations and map cognitive representations to cognitive representation of same kind. They are clearly analogous to morphisms in category theory.

2. M^8-H duality [1, 2, 7, 9] is a number theoretic analogue of momentum-position duality. M_c^8 serves as the analog of momentum space and $H=M^4\times CP_2$ as the analog of position space. M^8-H duality maps the 4-surface defined in M_c^8 by number theoretic holography based on 3-D data to a 4-D space-time surface in H.

3. Should M^8-H duality respect the algebraic extension? If so, it would map the cognitive representation defined by points belonging to 4-D surface $Y^4\subset M^8$ with the values of preferred coordinates in E to points of $M^4\subset H$ with coordinate values in E. One could say that M^8-H duality respects the number theoretical character of cognitive representations. The precise meaning of this intuition is however far from clear.

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There are also questions related to the choice of preferred coordinates in which the cognitive representation is defined.

1. Number theoretic constraints fix the preferred coordinates at M^8 side rather uniquely and this induces a preferred choice also on $M^4 \subset H$. For hyperbolic spaces (mass shells) a cognitive explosion happens and a natural question whether cognitive explosion happens also for the light-like curves assignable to the partonic orbits. If the light-like curve is geodesic, the explosion indeed occurs. For more general light-like curves this is not the case always: could these more general light-like curves be related by a birational map to light-like geodesics?

2. At the H side one can also imagine besides standard Minkowski coordinates also other physically preferred choices of coordinates: are they also theoretically preferred? The notion of Hamilton-Jacobi structure [6] suggests that in the case of M^4 Hamilton-Jacobi coordinates are very natural for the holomorphic realization of holography. If these are allowed, a natural condition would be that the Hamilton-Jacobi coordinates are related to each other by birational maps mapping the point of E to points of E so that cognitive representations are mapped to cognitive representations.

2 $M^8 - H$ duality, holography as holomorphy, Hamilton-Jacobi structures, and birational maps as cognitive morphisms

In the sequel the questions raised in the introduction are considered. The basic notions are $M^8 - H$ duality [1, 2, 7, 9], holography as a generalized holomorphy [5, 8], Hamilton-Jacobi structures [6], and birational maps as cognitive morphisms.

2.1 About more precise definitions of the basic concepts

Consider first more precise definitions of various notions involved.

1. What are the preferred coordinates of M^8_c in which the cognitive representation is constructed? M^8_c has a number theoretic interpretation in terms of complexified octonions and physical interpretation as 8-D momentum space. Linear Minkowski coordinates are number-theoretically preferred since octonionic multiplication and other arithmetic operations have a very simple form in these coordinates. Also the number theoretic automorphisms respect the arithmetic operations. The allowed automorphisms correspond to the group G_2 which is a subgroup of $SO(1,7)$. Physically Minkowski space coordinates are preferred coordinates in the momentum space and also in $M^4 \subset H$.

2. How the algebraic extension of rationals, call it E, is determined? The proposal is that rational polynomials characterize partially the 3-D data for number theoretic holography [7]. The roots of a rational polynomial P define an algebraic extension of rationals, call it E. A stronger, physically motivated, condition on P is that its coefficients are integers smaller than the degree of P.

   The roots of P define mass shells $H^3_c \subset M^4_c \subset M^8_c$, which in turn assign to the mass shells a 4-D surface Y^4 of M^8_c going through the mass shells by associative holography requiring that the normal space of Y^4 is associative, that is quaternionic. It has been be assumed that the roots are complex although also the condition that the roots are real can be considered. The imaginary unit i associated with the roots commutes with the octonionic imaginary units.

3. How the cognitive representation is defined? The points of $Y^4 \subset M^8_c$ with M^4 coordinates in E define a unique discretization of Y^4, called a cognitive representation, making sense also in the extensions of p-adic number fields induced by E. In general, the number of algebraic points in the interior of Y^4 is discrete and even finite but at the mass shells H^3 a cognitive explosion takes place. All points of H^3 with coordinates in E are algebraic.

The algebraic points with coordinates, which are algebraic *integers* are physically and cognitively in very special role in number theoretic physics and make sense also as points of various p-adic number fields making possible number theoretical universality. The points of H^3 have interpretation as momenta and for physical states the total momentum as sum of momenta at mass shells defined by the roots of P has components which are integers, called Galois confinement [3, 4], would define fundamental mechanism for the formation of bound states.

4. $M^8 - H$ duality maps the points of $H_c^3 \subset M_c^4 \subset M_c^8$ to points of $H^3 \subset M^4 \subset M^4 \times CP_2 = H$ by a map, which is essentially an inversion: this form is motivated by Uncertainty Principle: for the most recent formulation of the duality see [9]. This map is a birational map and takes points of E points of E. Also the points of cognitive representation belonging to the interior of $Y^4 \subset M_c^8$ are mapped to the interior of $X^4 \subset M_c^8$. One can ask whether the discrete set of points of cognitive representations in the interiors are of special physical interest, say having interpretation as interaction vertices.

2.2 Questions to be pondered

There are many questions to be considered.

1. Also partonic orbits in $X^4 \subset H$ define 3-D holographic data in H. What are these partonic orbits? The simplest partonic orbits have light-like M^4 projection but also more general light-like H projection can be considered (note the analogy with a 2-D rigid body rotating along a light-like geodesic of H). A general light-like geodesic of H is a combination of time-like geodesic of M^4 and space-like geodesic of CP_2.

   A point of the light-like partonic orbit correspond at the level of M^8 to the 3-D blowup of a point of M^8 at which the quaternionic normal space parametrized by CP_2 point is not unique so that the normal spaces for a 3-D section of CP_2, whose union along (probably light-like) geodesic is CP_2 with two holes corresponding to the ends of the partonic orbit. This singularity is highly analogous to the singularity of the electric field of a point charge. Partonic orbits define part of the 3-D holographic data.

2. Could one associate cognitive representations also to the partonic orbits? Could also partonic orbits allow a cognitive explosion? The simplest way to guarantee light-likeness for the H projection is as a light-like geodesic and this indeed allows an infinite number of algebraic points in Minkowski coordinates. Same applies to more general light-like orbits. One would have at least 1-D explosion of the cognitive representation.

3. What can one say about the CP_2 and M^4 projections of the partonic 2-surface? Could also these projections to X^2 and Y^2 allow an infinite number of points with coordinates in E or do these kinds of points have some special physical meaning, say as vertices for particle reactions? Concerning cognitive representation, the blow-up would mean that the point has an infinite but discrete set of quaternionic normal spaces at the level of M^8. Since the partonic surface can have an arbitrary complex sub-manifold as CP_2, there is indeed information to be cognitively represented.

2.3 The most general cognitively preferred coordinate choices for space-time surfaces and H?

In the case of M_c^8, number theoretical considerations fix the preferred coordinates highly uniquely. In the case of H the situation is not so obvious and one cannot exclude alternative coordinate choices related by a birational map.

A possible motivation comes from the following argument.

1. String world sheets are candidates for singularities analogous to partonic orbits. At a given point of the string world sheet a blow up to a 2-D complex sub-manifold of CP_2 would occur. This would mean that the normal spaces of the point in M_c^8 form this sub-manifold. Cosmic strings are the simplest objcts of this kind. Monopole flux tubes are deformations of the cosmic strings and allow also an interpretation in terms of maps from M^4 to CP_2.

   If string world sheets define part of the data needed to define holography, one could argue that it makes sense to assign cognitive explosion to the string world sheet.

2. Cognitive explosion takes place if the string world sheets are 2-D geodesic submanifolds of H. Planes $M^2 \subset M^4$ represent the simplest example. A more complex example is obtained by taking a space-like geodesic in H and rotating it along a time-like geodesic of H. One can also take a light-like geodesic in H and rotate it along a light-like geodesic in dual light-like direction (ruler surface would be in question). In which case the gluing of the string world sheet along the boundary to the partonic orbit could be possible.

   One might perhaps think of building string world sheets by gluing these kinds of ultrasimple regions along their boundaries so that one would have edges. An interpretation as a discretization would be appropriate. One might even go further and argue that the cognitive explosion explains why we are able to think of these kinds of regions in terms of simple formulas. One might argue that number theoretic physics realizes exactly what is usually regarded as approximation. One can however wonder whether life is so simple.

This argument encourages to consider a more complex option allowing more general string world sheets.

1. In the case of M^4 projection, the notion of the Hamilton-Jacobi structure [6], generalizing the notion of ordinary complex structure, is highly interesting in this respect. It involves a generalization of complex coordinates involving local decompositions $M^4(x) = M^2(x) \times E^2(x)$ of the 4-D tangent space of M^4. The integrable distribution of $E^2(x)$ corresponds to complex coordinates $(w, \overline{w}$ integrating to a partonic 2-surface whereas the integrable distribution of $M^2(x)$ to light-like coordinate pairs (u, v) integrating to a string world sheet in M^4.

   Cognitive representation mean that the discretized values of the Hamilton-Jacobi coordinates $(u, v, w, \overline{w})$ are in E. Hamilton-Jacobi structure generalizes also to the level of $X^4 \subset H$ and now Y^2 can also correspond to CP_2 projection as in the case of cosmic strings and magnetic flux tubes. Note that in TGD one can use a subset of H coordinates as coordinates of X^4.

2. The simplest assumption is that the 1-D parton orbit corresponds to a light-like geodesic but could one map light-like geodesics to more general light-like curves by a birational map? Hamilton-Jacobi structure gives rise to a pair of curved (u, v) of light-like coordinates: could it relate to the standard flat light-like coordinates of M^2 by a birational map? Could a birational map relate standard complex coordinates of E^2 to the pair $(w, \overline{w})$? Could one also consider more general birational maps of $M^4 \to M^4$? If so, the Hamilton-Jacobi structures would be related by maps respecting algebraic extensions and cognitive representations. This would give a very powerful constraint on the Hamilton-Jacobi structures.

In the case of CP_2, projective coordinates are group-theoretically highly unique and determined apart from color rotations. Could one require that the CP_2 projection Y^2 associated with the partonic 2-surface and cosmic string or magnetic flux tube involves cognitive explosion. Are the allowed M^4 and CP_2 projections related by birational maps? Note that color rotations are birational maps.

These considerations suggest the following speculative view.

1. $M^8 - H$ duality, when restricted to 3-D holographic data at both sides, is analogous to a birational map expressible in terms of rational functions and respects the number theoretical character of cognitive representations.

2. Cognitive explosion occurs for the holographic data (this is very natural from the information theoretic perspective): this includes also string world sheets. Hamilton-Jacobi structures in the same cognitive class, partially characterized by the extension E of rationals, are related by a birational map.

3. M^8-H duality maps the quaternionic normal spaces to points of CP_2 and is an example of a birational map in M^4 degrees of freedom. It is not however easy to guess how the number theoretic holography is realized explicitly and how the 4-surfaces in M^8 are mapped to holomorphic 4-surfaces in H.

4. An interesting additional aspect relates to the non-determinism of partonic orbits due to the non-determinism of the light-likeness condition deriving from the fact that the action is Chern-Simons-Kähler action. The deformation of the partonic orbit induces the deformation of time derivatives of H coordinates at the boundary of $\delta M^4_+ \times CP_2$ to guarantee that boundary conditions at the orbit are realized. This suggests a strong form of holography [8]. Already the 3-surfaces at $\delta M^4_+ \times CP_2$ or partonic orbits would be enough as holographic data. This in turn suggests that the analog of birational cognitive correspondence between the holographic data at $\delta M^4_+ \times CP_2$ and at partonic orbits.

3 Appendix: Some facts about birational geometry

Birational geometry has as its morphisms birational maps: both the map and its inverse are expressible in terms of rational functions. The coefficients of polynomials appearing in rational functions are in the TGD framework rational. They map rationals to rationals and also numbers of given extension E of rationals to themselves (one can assign to each space-time region an extension defined by a polynomial).

Therefore birational maps map cognitive representations, defined as discretizations of the space-time surface such that the points have physically/number theoretically preferred coordinates in E, to cognitive representations. They therefore respect cognitive representations and are morphisms of cognition. They are also number-theoretically universal, making sense for all p-adic number fields and their extensions induced by E. This makes birational maps extremely interesting from the TGD point of view.

The following lists basic facts about birational geometry as I have understood them on the basis of Wikipedia articles about birational geometry and Enriques-Kodaira classification. I have added physics inspired associations with TGD.

Birational geometries are one central approach to algebraic geometry.

1. They provide classification of complex varieties to equivalence classes related by birational maps. The classification complex curves (real dimension 2) reduces to the classification of projective curves of projective space CP_n determined as zeros of a homogeneous polynomial. Complex surfaces (real dimension 4) are of obvious interest in TGD: now however the notion of complex structure is generalized and one has Hamilton-Jacobi structure.

2. In TGD, a generalization of complex surfaces of complex dimension 2 in the embedding space $H=M^4\times CP_2$ of complex dimension 4 is considered. What is new is the presence of the Minkowski signature requiring a combination of hypercomplex and complex structures to the Hamilton-Jacobi structure. Note however the space-time surfaces also have counterparts in the Euclidean signature $E^4\times CP_2$: whether this has a physical interpretation, remains an open question. Second representation is provided as 4-surfaces in the space M^8_c of complexified octonions and an attractive idea is that M^8-H duality corresponds to a birational mapping of cognitive representations to cognitive representations.

3. Every algebraic variety is birationally equivalent with a sub-variety of CP_n so that their classification reduces to the classification of projective varieties of CP_n defined in terms of homogeneous

polynomials. $n = 2$ (4 real dimensions) is of special relevance from the TGD point of view. A variety is said to be rational if it is birationally equivalent to some projective variety: for instance CP_2 is rational.

4. A concrete example of birational equivalence is provided by stereographic projections of quadric hypersurfaces in n+1-D linear space. Let p be a point of quadric. The stereographic projection sends a point q of the quadric to the line going through p and q, that is a point of CP_n in the complex case. One can select one point on the line as its representative. Another exammple is provided by Möbius transformations representing Lorentz group as transformations of complex plane.

The notion of a minimal model is important.

1. The basic observation is that it is possible to eliminate or add singularities by using birational maps of the space in which the surface is defined to some other spaces, which can have a higher dimension. The zeros of a birational map can be used to eliminate singularities of the algebraic surface of dimension n by blowups replacing the singularity with CP_n. Poles in turn create singularities. Peaks and self-intersections are examples of singularities.

   The idea is to apply birational maps to find a birationally equivalent surface representation, which has no singularities. There is a very counter-intuitive formal description for this. For instance, complex curves of CP_2 have intersections since their sum of their real dimensions is 4. The same applies to 4-surfaces in H. My understanding is as follows: the blowup for CP_2 makes it possible to get rid of an intersection with intersection number 1. One can formally say that the blow up by gluing a CP_1 defines a curve which has negative intersection number -1.

2. In the TGD framework, wormhole contacts have the same metric and Kähler structure as CP_2 and light-like M^4 projection (or even H projection). They appear as blowups of singularities of 4-surfaces along a light-like curve of M^8. The union of the quaternionic/associative normal spaces along the curve is not a line of CP_2 but CP_2 itself with two holes corresponding to the ends of the light-like curve. The 3-D normal spaces at the points of the light-like curve are not unique and form a local slicing of CP_2 by 3-D surfaces. This is a Minkowskian analog of a blow-up for a point and also an analog of cut of analytic function.

The Italian school of algebraic geometry has developed a rather detailed classification of these surfaces. The main result is that every surface X is birational either to a product $CP_1 \times C$ for some complex curve C or to a minimal surface Y. Preferred extremals are indeed minimal surfaces so that space-time surfaces might define minimal models. The absence of singularities (typically peaks or self-intersections) characterizing minimal models is indeed very natural since physically the peaks do not look acceptable.

There are several birational invariants listed in the Wikipedia article. Many of them are rather technical in nature. The canonical bundle K_X for a variety of complex dimension n corresponds to n:th exterior power of complex cotangent bundle that is holomorphic n-forms. For space-time surfaces one would have $n = 2$ and holomorphic 2-forms.

1. Plurigenera corresponds to the dimensions for the vector space of global sections $H_0(X, K_X^d)$ for smooth projective varieties and are birational invariants. The global sections define global coordinates, which define birational maps to a projective space of this dimension.

2. Kodaira dimension measures the complexity of the variety and characterizes how fast the plurigenera increase. It has values $-\infty, 0, 1, ..n$ and has 4 values for space-time surfaces. The value $-\infty$ corresponds to the simplest situation and for $n = 2$ characterizes CP_2 which is rational and has vanishing plurigenera.

3. The dimensions for the spaces of global sections of the tensor powers of complex cotangent bundle (holomorphic 1-forms) define birational invariants. In particular, holomorphic forms of type $(p, 0)$

are birational invariants unlike the more general forms having type (p, q). Betti numbers are not in general birational invariants.

4. Fundamental group is birational invariant as is obvious from the blowup construction. Other homotopy groups are not birational invariants.

5. Gromow-Witten invariants are birational invariants. They are defined for pseudo-holomorphic curves (real dimension 2) in a symplectic manifold X. These invariants give the number of curves with a fixed genus and 2-homology class going through n marked points. Gromow-Witten invariants have also an interpretation as symplectic invariants characterizing the symplectic manifold X.

   In TGD, the application would be to partonic 2-surfaces of given genus g and homology charge (Kähler magnetic charge) representatable as holomorphic surfaces in $X = CP_2$ containing n marked points of CP_2 identifiable as the loci of fermions at the partonic 2-surface. This number would be of genuine interest in the calculation of scattering amplitudes.

What birational classification could mean in the TGD framework?

1. Holomorphic ansatz gives the space-time surfaces as Bohr orbits. Birational maps give new solutions from a given solution. It would be natural to organize the Bohr orbits to birational equivalence classes, which might be called cognitive equivalence classes. This should induce similar organization at the level of M^8_c.

2. An interesting possibility is that for certain space-time surfaces CP_2 coordinates can be expressed in terms of preferred M^4 coordinates using birational functions and vice versa. Cognitive representation in M^4 coordinates would be mapped to a cognitive representation in CP_2 coordinates.

3. The interpretation of $M^8 - H$ duality as a generalization of momentum position duality suggests information theoretic interpretation and the possibility that it could be seen as a cognitive/birational correspondence. This is indeed the case M^4 when one considers linear M^4 coordinates at both sides.

4. An intriguing question is whether the pair of hypercomplex and complex coordinates associated with the Hamilton-Jacobi structure could be regarded as cognitively acceptable coordinates. If Hamilton-Jacobi coordinates are cognitively acceptable, they should relate to linear M^4 coordinates by a birational correspondence so that $M^8 - H$ duality in its basic form could be replaced with its composition with a coordinate transformation from the linear M^4 coordinates to particular Hamilton-Jacobi coordinates. The color rotations in CP_2 in turn define birational correspondences between different choices of Eguchi-Hanson coordinates.

   If this picture makes sense, one could say that the entire holomorphic space-time surfaces, rather than only their intersections with mass shells H^3 and partonic orbits, correspond to cognitive explosions. This interpretation might make sense since holomorphy has a huge potential for generating information: it would make TGD exactly solvable.

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References

[1] Pitkänen M. A critical re-examination of $M^8 - H$ duality hypothesis: part I. Available at: https://tgdtheory.fi/public_html/articles/M8H1.pdf., 2020.

[2] Pitkänen M. A critical re-examination of $M^8 - H$ duality hypothesis: part II. Available at: https://tgdtheory.fi/public_html/articles/M8H2.pdf., 2020.

[3] Pitkänen M. About TGD counterparts of twistor amplitudes: part I. https://tgdtheory.fi/public_html/articles/twisttgd1.pdf., 2022.

[4] Pitkänen M. About TGD counterparts of twistor amplitudes: part II. https://tgdtheory.fi/public_html/articles/twisttgd2.pdf., 2022.

[5] Pitkänen M. About the Relationships Between Weak and Strong Interactions and Quantum Gravity in the TGD Universe . https://tgdtheory.fi/public_html/articles/SW.pdf., 2023.

[6] Pitkänen M. Holography and Hamilton-Jacobi Structure as 4-D generalization of 2-D complex structure. https://tgdtheory.fi/public_html/articles/HJ.pdf., 2023.

[7] Pitkänen M. New findings related to the number theoretical view of TGD. https://tgdtheory.fi/public_html/articles/M8Hagain.pdf., 2023.

[8] Pitkänen M. Symmetries and Geometry of the "World of Classical Worlds" . https://tgdtheory.fi/public_html/articles/wcwsymm.pdf., 2023.

[9] Pitkänen M. The TGD view of the recently discovered gravitational hum as gravitational diffraction. https://tgdtheory.fi/public_html/articles/gravhum.pdf., 2023.