Article

Potentials for the Lanczos Spintensor in Gödel Spacetime

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Abstract
For the Gödel geometry we obtain several potentials for the Lanczos generator associated with the Weyl tensor.

Keywords: Lanczos potential, Gödel's metric, conformal tensor.

1. Introduction

The Lanczos potential K_{abc} has the properties [1, 2]:

$$K_{abc} = -K_{bac}\,, \qquad K_{abc} + K_{bca} + K_{cab} = 0, \qquad K_a{}^b{}_b = 0, \tag{1}$$

and for the Gödel's geometry [3-9]:

$$ds^2 = (dx^0)^2 + 2\, e^{x^3} dx^0\, dx^1 + \tfrac{1}{2}\, e^{2x^3} (dx^1)^2 - (dx^2)^2 - (dx^3)^2, \tag{2}$$

we have the expression [10-16]:

$$S_{abc} \equiv K_{abc} + i\, {}^*K_{abc} = \tfrac{i}{9}\, [U_{ab}\, m_c + V_{ab}\, \overline{m}_c - M_{ab}\, (l_c + n_c)], \tag{3}$$

with the canonical null tetrad [17]:

$$(l^a) = \frac{1}{\sqrt{2}}(1, 0, -1, 0), \qquad (n^a) = \frac{1}{\sqrt{2}}(1, 0, 1, 0), \qquad (m^a) = \left(1, -e^{-x^3}, 0, \frac{i}{\sqrt{2}}\right),$$
$$(\overline{m}^a) = \left(1, -e^{-x^3}, 0, -\tfrac{i}{\sqrt{2}}\right), \tag{4}$$

and:

$$V_{ab} = l_a\, x\, m_b\,, \qquad U_{ab} = \overline{m}_a\, x\, n_b\,, \qquad M_{ab} = m_a\, x\, \overline{m}_b + n_a\, x\, l_b\,. \tag{5}$$

The corresponding expression for the conformal tensor is given by [6, 18]:

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$$S_{abcr} \equiv C_{abcr} + i\ {}^{*}C_{abcr} = -\tfrac{1}{3}(M_{ab}\, M_{cr} + V_{ab}\, U_{cr} + U_{ab}\, V_{cr}), \qquad (6)$$

with the following non-zero Newman-Penrose (NP) quantities [19-21]:

$$\mu = \rho = 2\gamma = 2\varepsilon = \sqrt{2}\,\alpha = \sqrt{2}\,\beta = \tfrac{i}{2}, \psi_2 = -\tfrac{1}{6}, \quad \phi_{00} = \phi_{22} = 2\phi_{11} = \tfrac{1}{4}, \quad R = -1, \qquad (7)$$

and from (3) it is clear that the NP components of the Lanczos generator are:

$$\Omega_1 = \Omega_6 = -\tfrac{i}{18}, \qquad \Omega_r = 0, \quad r \neq 1, 6, \qquad (8)$$

which gives the following components different to zero:

$$K_{031} = K_{103} = \tfrac{1}{2}\, K_{130} = \tfrac{1}{18}\, e^{x^3}, \qquad K_{131} = \tfrac{1}{6}\, e^{2x^3},$$

$${}^{*}K_{233} = \tfrac{1}{2}\ \ {}^{*}K_{200} = \tfrac{\sqrt{2}}{18}, \qquad {}^{*}K_{201} = {}^{*}K_{210} = \tfrac{\sqrt{2}}{9}\, e^{x^3}, \qquad {}^{*}K_{211} = \tfrac{5\sqrt{2}}{36}\, e^{2x^3},$$

$$R_{0303} = \tfrac{1}{2}, \ R_{0313} = \tfrac{1}{2}\, e^{x^3}, \ R_{1313} = 3\, R_{0101} = \tfrac{3}{4} e^{2x^3}, R_{00} = -1, \ R_{01} = -e^{x^3}, \ R_{11} = -e^{2x^3},$$

$$C_{0303} = -C_{2323} = -\tfrac{1}{2} C_{0202} = \tfrac{1}{6}, \ C_{0313} = \tfrac{1}{2} C_{1220} = \tfrac{1}{6} e^{x^3}, \ C_{0101} = \tfrac{1}{4} C_{1313} = -\tfrac{1}{5} C_{1212} = \frac{e^{2x^3}}{12}, \qquad (9)$$

$$\left(e_{(0)}{}^{a}\right) \equiv \left(\tfrac{1}{\sqrt{2}}(l^a + n^a)\right) = (1,0,0,0), \qquad \left(e_{(1)}{}^{a}\right) \equiv \left(\tfrac{1}{\sqrt{2}}(m^a + \overline{m}^a)\right) = \left(\sqrt{2}, -\sqrt{2}\, e^{-x^3}, 0, 0\right),$$

$$\left(e_{(2)}{}^{a}\right) \equiv \left(\tfrac{i}{\sqrt{2}}(m^a - \overline{m}^a)\right) = (0,0,0,-1), \left(e^{a}_{(3)}\right) \equiv \left(\tfrac{1}{\sqrt{2}}(l^a - n^a)\right) = (0,0,-1,0), \Gamma^{3}{}_{11} = \frac{e^{2x^3}}{2}$$

$$\Gamma_{013} = \Gamma_{103} = -\Gamma_{301} = \tfrac{1}{2}\, e^{x^3}, \quad \Gamma_{113} = -\Gamma_{311} = \tfrac{1}{2}\, e^{2x^3}, \quad \Gamma^{3}{}_{01} = \Gamma^{0}{}_{13} = \tfrac{1}{2}\, e^{x^3}, \Gamma^{0}{}_{03} = 1,$$

$$\Gamma^{1}{}_{03} = -e^{-x^3}.$$

We note that $\left(e_{(0)}{}^{b}\right)$ is a Killing time-like vector and $\left(e_{(3)}{}^{b}\right)$ is a constant space-like vector, that is:

$$e_{(0)\ a;b} + e_{(0)\ b;a} = 0, \qquad e_{(3)\ a;b} = 0. \qquad (10)$$

In Sec. 2 we employ all above information to construct potentials for the Lanczos spintensor in according with the fundamental relation [1]:

$$C_{abcd} = K_{abc;d} - K_{abd;c} + K_{cda;b} - K_{cdb;a} + K_{ad}\, g_{bc} - K_{ac}\, g_{bd} + K_{bc}\, g_{ad} - K_{bd}\, g_{ac}, \quad (11)$$

where:

$$K^a{}_b = K_b{}^a = K^{ac}{}_{b;c}\,, \qquad K^{abc}{}_{;c} = 0, \quad (12)$$

with the non-zero components:

$$-\tfrac{1}{2}\, K^0{}_0 = K^1{}_1 = K^3{}_3 = \tfrac{1}{6}, \qquad K^0{}_1 = -\tfrac{1}{2}\, e^{x^3}. \quad (13)$$

2. Several generators for the Lanczos potential

Now we study if the following relation has solution for some constant *q*:

$$q\, K^{abc} = C^{abcr}{}_{;r}\ , \quad (14)$$

which is equivalent to a set of eight differential equations in the NP formalism:

$$\begin{aligned}
q\,\Omega_0 &= D\psi_1 - \bar{\delta}\psi_0 + (4\alpha - \pi)\,\psi_0 - 2(2\rho + \varepsilon)\,\psi_1 + 3\kappa\,\psi_2,\\
q\,\Omega_1 &= D\psi_2 - \bar{\delta}\psi_1 + \lambda\,\psi_0 + 2(\alpha - \pi)\,\psi_1 - 3\rho\,\psi_2 + 2\kappa\,\psi_3,\\
q\,\Omega_2 &= D\psi_3 - \bar{\delta}\psi_2 + 2\lambda\,\psi_1 - 3\pi\,\psi_2 + 2(\varepsilon - \rho)\,\psi_3 + \kappa\,\psi_4,\\
q\,\Omega_3 &= D\psi_4 - \bar{\delta}\psi_3 + 3\lambda\,\psi_2 - 2(2\pi + \alpha)\,\psi_3 + (4\varepsilon - \rho)\,\psi_4,\\
q\,\Omega_4 &= \delta\psi_1 - \Delta\psi_0 + (4\gamma - \mu)\,\psi_0 - 2(2\tau + \beta)\,\psi_1 + 3\sigma\,\psi_2,\\
q\,\Omega_5 &= \delta\psi_2 - \Delta\psi_1 + \nu\,\psi_0 + 2(\gamma - \mu)\,\psi_1 - 3\tau\,\psi_2 + 2\sigma\,\psi_3,\\
q\,\Omega_6 &= \delta\psi_3 - \Delta\psi_2 + 2\nu\,\psi_1 - 3\mu\,\psi_2 + 2(\beta - \tau)\,\psi_3 + \sigma\,\psi_4,\\
q\,\Omega_7 &= \delta\psi_4 - \Delta\psi_3 + 3\nu\,\psi_2 - 2(2\mu + \gamma)\,\psi_3 + (4\beta - \tau)\,\psi_4,
\end{aligned} \quad (15)$$

where we can employ the values (7) and (8) to obtain $q = -\frac{9}{2}$, thus (14) implies the expression:

$$K^{abc} = -\frac{2}{9}\, C^{abcr}{}_{;r}\,, \tag{16}$$

which may be considered as the inversion of (11), that is, the Weyl tensor is a generator of the Lanczos potential in the Gödel's cosmological model.

The next stage is to search solutions for the tensor equation:

$$K_{abc} = P_{ca;b} - P_{cb;a}\,, \qquad P_{jr} = P_{rj}\,, \qquad P = P^{r}{}_{r}\,, \tag{17}$$

via its corresponding NP equations:

$$\Omega_0 = 2[D\Omega_{01} - \delta\Omega_{00} + (2\bar{\alpha} + 2\beta - \bar{\pi})\,\Omega_{00} - 2(\bar{\rho} + \varepsilon)\,\Omega_{01} - 2\sigma\,\Omega_{10} + \bar{\kappa}\,\Omega_{02} + 2\kappa\,\Omega_{11}],$$

$$\Omega_1 = 2\left[D\Omega_{11} - \delta\Omega_{10} + \tfrac{1}{8}DP + \mu\,\Omega_{00} + (2\bar{\alpha} - \bar{\pi})\,\Omega_{10} - \pi\,\Omega_{01} - \sigma\,\Omega_{20} - 2\bar{\rho}\,\Omega_{11} + \kappa\,\Omega_{21} + \bar{\kappa}\,\Omega_{12}\right],$$

$$\Omega_2 = 2\left[\bar{\delta}\Omega_{11} - \Delta\Omega_{10} - \tfrac{1}{8}\bar{\delta}P + \nu\,\Omega_{00} + (2\bar{\gamma} - \bar{\mu})\,\Omega_{10} - \lambda\,\Omega_{01} - \tau\,\Omega_{20} - 2\bar{\tau}\,\Omega_{11} + \rho\,\Omega_{21} + \sigma\,\Omega_{12}\right],$$

$$\Omega_3 = 2[\bar{\delta}\Omega_{21} - \Delta\Omega_{20} + 2\nu\,\Omega_{10} + (2\bar{\gamma} - 2\gamma - \bar{\mu})\,\Omega_{20} - 2\lambda\,\Omega_{11} + 2(\alpha - \bar{\tau})\,\Omega_{21} + \bar{\sigma}\,\Omega_{22}],$$

$$\Omega_4 = 2[D\Omega_{02} - \delta\Omega_{01} + \bar{\lambda}\,\Omega_{00} + 2(\beta - \bar{\pi})\,\Omega_{01} + (2\bar{\varepsilon} - 2\varepsilon - \bar{\rho})\,\Omega_{02} - 2\sigma\,\Omega_{11} + 2\kappa\,\Omega_{12}], \tag{18}$$

$$\Omega_5 = 2\left[D\Omega_{12} - \delta\Omega_{11} + \tfrac{1}{8}\delta P + \mu\,\Omega_{01} + \bar{\lambda}\,\Omega_{10} - \pi\,\Omega_{02} - 2\bar{\pi}\,\Omega_{11} + (2\bar{\varepsilon} - \bar{\rho})\,\Omega_{12} - \sigma\,\Omega_{21} + \kappa\,\Omega_{22}\right],$$

$$\Omega_6 = 2\left[\bar{\delta}\Omega_{12} - \Delta\Omega_{11} - \tfrac{1}{8}\Delta P + \nu\,\Omega_{01} + \bar{\nu}\,\Omega_{10} - \lambda\,\Omega_{02} - 2\bar{\mu}\,\Omega_{11} + \left(2\bar{\beta} - \bar{\tau}\right)\Omega_{12} - \tau\,\Omega_{21} + \rho\,\Omega_{22}\right],$$

$$\Omega_7 = 2[\bar{\delta}\Omega_{22} - \Delta\Omega_{21} + \bar{\nu}\,\Omega_{20} + 2\nu\,\Omega_{11} - 2(\bar{\mu} + \gamma)\,\Omega_{21} - 2\lambda\,\Omega_{12} + \left(2\alpha + 2\bar{\beta} - \bar{\tau}\right)\Omega_{22}],$$

$$\Omega_1 + \bar{\Omega}_1 = 2\left[D\Omega_{11} - \Delta\Omega_{00} - \tfrac{1}{8}DP + 2(\gamma + \bar{\gamma})\Omega_{00} - (2\bar{\tau} + \pi)\Omega_{01} - (2\tau + \bar{\pi})\Omega_{10} + \bar{\kappa}\Omega_{12} + \kappa\Omega_{21}\right],$$

$$\overline{\Omega}_2 - \Omega_5 = 2\left[\delta\Omega_{11} - \bar{\delta}\Omega_{02} + \tfrac{1}{8}\delta P + (2\bar{\mu} - \mu)\,\Omega_{01} - \bar{\lambda}\,\Omega_{10} + 2(\alpha - \bar{\beta})\,\Omega_{02} + (\bar{\rho} - 2\rho)\,\Omega_{12} + \sigma\,\Omega_{21}\right],$$

$$\Omega_6 + \overline{\Omega}_6 = 2\left[D\Omega_{22} - \Delta\Omega_{11} + \tfrac{1}{8}DP + \nu\Omega_{01} + \bar{\nu}\Omega_{10} - (\bar{\tau} + 2\pi)\Omega_{12} - (\tau + 2\bar{\pi})\Omega_{21} + 2(\varepsilon + \bar{\varepsilon})\Omega_{22}\right],$$

where $\Omega_{ab} = \overline{\Omega}_{ba}$ are the NP components of $Q_{ab} = P_{ab} - \frac{P}{4}\, g_{ab}$. It is easy to see that a solution of (18) is $P_{ab} = -\frac{1}{9}\, R_{ab}$, that is, $\Omega_{ab} = -\frac{1}{9}\, \phi_{ab}$ and $P = -\frac{1}{9}\, R = \frac{1}{9}$, therefore (17) gives the relation:

$$K_{abc} = -\tfrac{1}{9}\left(R_{ca;b} - R_{cb;a}\right), \tag{19}$$

which is compatible with (16) due to the Bianchi identities [5, 6, 21]; hence the Ricci tensor also is a generator of the Lanczos spintensor in Gödel spacetime.

With (9) is possible to deduce the wave equation for some quantities of interest, for example:

$$\Box K^{abc} \equiv K^{abc;r}{}_{;r} = -3\, K^{abc}, \qquad \Box R_{ab} = 2\,\Box K_{ab} = -6\, K_{ab},\ \Box e_{(0)a} = -e_{(0)a}\ , \tag{20}$$

besides:

$$K_{ab;c} + K_{bc;a} + K_{ca;b} = 0,\ K^{ab}{}_{;b} = 0, R_{abcr}\, K^{abc} = 0,$$

$$R^{r}{}_{[a}K_{b]cr} - R^{r}{}_{c}K_{abr} - R^{r}{}_{q}K_{r[a}{}^{q}g_{b]c} = K_{abc}, \tag{21}$$

$$K_{ab} = -\tfrac{1}{6}\left(2\, e_{(0)a}\, e_{(0)b} + e_{(1)a}\, e_{(1)b} + e_{(2)a}\, e_{(2)b}\right) = \tfrac{1}{6}\left(R_{ab} - 2R_{acrb}\, G^{cr}\right),$$

and K_{ab} turns out to be a potential for the Lanczos tensor:

$$K_{abc} = -\tfrac{2}{9}\,\left(K_{ca;b} - K_{cb;a}\right), \tag{22}$$

thus (22) is the inversion of (12).

An adequate linear combination of R_{ab} and K_{ac} gives an interesting generator:

$$K_{abc} = -\tfrac{\sqrt{2}}{18}\,\left(B_{ca;b} - B_{cb;a}\right), \qquad B^{ac}{}_{;c} = 0,$$

$$B_{ac} = \tfrac{1}{\sqrt{2}}\left(6K_{ac} - R_{ac}\right) = \tfrac{1}{\sqrt{2}}\left(2R_{ac} + g_{ac} + e_{(3)a}e_{(3)c}\right) = \sqrt{2}\left(G_{ac} + \tfrac{1}{2}e_{(3)a}e_{(3)c}\right), \tag{23}$$

$$\Box B_{ac} = -6\sqrt{2}\, K_{ac},$$

where $G_{ab} = R_{ab} - \frac{R}{2} g_{ab}$ is the Einstein tensor, and it is remarkable the fulfillment of the Gauss equation:

$$R_{abcr} = B_{ac}\, B_{br} - B_{ar}\, B_{bc},$$
$$B_{ac} = -\sqrt{2}\, R_{abrc}\, G^{br}\,, \qquad {}^{*}R^{*abcr}\, R_{abcr} = 0, \tag{24}$$

hence B_{ac} generates the Lanczos potential via the differential relation (23) and also the curvature tensor through the algebraic expression (24).

Finally, in [22] were obtained the following properties in Gödel geometry:

$$K_{abc} = \frac{\sqrt{2}}{3}\ {}^{*}C_{abcr}\, e_{(3)}{}^{r}\,, \qquad {}^{*}K_{abc} = -\frac{\sqrt{2}}{3}\, C_{abcr}\, e_{(3)}{}^{r}\,, \tag{25}$$

which indicate an algebraic connection between Lanczos and Weyl tensors, unlike the differential relationship (16).

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