Article

On the Langer Transformation

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Abstract

We exhibit a simple procedure to deduce the Langer and Bateman *et al*-Mavromatis transformations to map the Coulomb problem into the Morse and simple harmonic oscillators, respectively.

Keywords: Langer mapping, Morse potential, Coulomb interaction.

1. Introduction

The Schrödinger equation for the Coulomb potential is given by [1-3]:

$$\frac{1}{2}\left[\frac{d^2R}{dr^2}-\frac{l(l+1)}{r^2}\,R\right]+\frac{\alpha}{r}\,R=\beta\,R, \qquad l=0,\ldots,n-1, \tag{1}$$

where:

$$\alpha=\frac{Z\,q^2}{4\,\pi\,\varepsilon_0}\,, \qquad \beta=\frac{Z^2q^4}{32\,\pi^2\,\varepsilon_0^2\,n^2}\,, \qquad n\geq 1. \tag{2}$$

Here we exhibit how changes (1) under the mapping:

$$R=A\,g(u)\,\psi(u), \qquad r=B\,f(u), \qquad A,B \text{ are constants}, \tag{3}$$

that is, the independent variable r and the Coulomb radial wave function R are transformed trying to obtain a new Schrödinger-like equation for ψ in the variable u. Hence the aim is find f and g implying another potential of physical interest associated to ψ.

In Sec. 2 we show that our approach gives, in natural manner, the Langer transformation [4] which allows to relate [5] the Coulomb potential with the Morse interaction [1, 2, 6, 7] for the vibrational motion of a diatomic molecule. Besides, it is also possible to deduce the Bateman et al [8]-Mavromatis [9] mapping which has been investigated in the context of connecting (1) and oscillator systems.

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2. Mappings between Schrödinger equations

We apply (3) into (1) to obtain:

$$\frac{1}{2}\left[\psi'' + \left(2\frac{g'}{g} - \frac{f''}{f'}\right)\psi' + \left(\frac{g''}{g} - \frac{f''}{f'}\frac{g'}{g} - l(l+1)\frac{f'^2}{f^2}\right)\psi\right] + \alpha\, B\, \frac{f'^2}{f}\,\psi = \beta\, B^2\, f'^2\,\psi, \qquad (4)$$

such that $\psi' = \frac{d\psi}{du}$; then (4) may be a Schrödinger-like equation if into it we eliminate the coefficient of ψ', thus:

$$\frac{f''}{f'} = 2\,\frac{g'}{g} \qquad \therefore \qquad f' = c\, g^2, \qquad c = \text{constant}, \qquad (5)$$

and (4) acquires the structure:

$$\frac{1}{2}\left[\psi'' + \left(\frac{g''}{g} - 2\frac{g'^2}{g^2} - c^2\, l(l+1)\frac{g^4}{f^2}\right)\psi\right] + c^2\,\alpha\, B\,\frac{g^4}{f}\psi = c^2\,\beta\, B^2\, g^4\,\psi. \qquad (6)$$

Now (6) offers several options, for example, into it we can make the coefficient of $-c^2 l(l+1)\psi$ equals to one, that is:

$$f = g^2, \qquad (7)$$

then (5) implies the Langer transformation [4]:

$$f = e^{\,cu}, \qquad g = e^{\frac{c}{2}u}, \qquad (8)$$

and (6) takes the form:

$$-\frac{1}{2}\,\psi'' + \frac{\alpha\, B\, c^2}{2}\left(\frac{2\beta\, B}{\alpha}\, e^{2cu} - 2\, e^{cu}\right)\psi = -\frac{c^2}{2}\,(l+\tfrac{1}{2})^2\,\psi, \qquad (9)$$

which is very attractive because permits to introduce the Morse potential [1, 2, 6, 7] if we select the values $c = -a$ and $\frac{2\beta B}{\alpha} = 1$ with the notation $D = \frac{\alpha B c^2}{2}$, then from (2):

$$B = \frac{4\pi\varepsilon_0}{Zq^2}\, n^2\,, \qquad D = \frac{a^2}{2}\, n^2, \qquad a > 0, \qquad (10)$$

therefore (9) becomes the vibrational Schrödinger equation for the Morse interaction:

$$-\frac{1}{2}\,\psi'' + D(e^{-2au} - 2\, e^{-au})\,\psi = E\,\psi, \qquad E = -\frac{a^2}{2}(l+\tfrac{1}{2})^2\,, \qquad (11)$$

where a is a range parameter (associated with the width of the potential well) and D is the energy of dissociation (well-depth) [10, 11].

Finally, from (3), (8) and (10) we obtain [5]:

$$R = A\, e^{-\frac{a}{2}u}\, \psi(u), \qquad\qquad r = \frac{4\pi\varepsilon_0}{Zq^2}\, n^2\, e^{-au}, \tag{12}$$

where, by normalization, we can employ the value $A^2 = \frac{Zq^2}{4\pi\varepsilon_0 n(l+\frac{1}{2})}$.

Now we back to the equation (6) seeking an alternative to (7), for example, to make the coefficient of $c^2\alpha\beta\psi$ equals to one, that is:

$$f = g^4\,, \tag{13}$$

then (5) gives the Bateman et al [8]-Mavromatis [9] transformation:

$$f = \frac{c^2}{4}\, u^2\,, \qquad\qquad g = \sqrt{\frac{c}{2}u}\,, \tag{14}$$

and (6) adopts the form:

$$-\frac{1}{2}\left[\psi'' - \frac{(4l+1)\,(4l+3)}{4\,u^2}\,\psi\right] + \frac{c^4\beta B^2}{4}\, u^2\,\psi = c^2\alpha\, B\,\psi\,, \tag{15}$$

being achieved the mapping of the hydrogen-like atom into the 3-dimensional simple harmonic oscillator with certain parameters.

It is clear that in (6) we can try different connections between f and g, respecting the constraint (5), to deduce Schrödinger-like equations associated to several potentials.

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