Article

On Stirling Numbers of the Second Kind

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Abstract

We give an elementary proof of Roman's identity between the Stirling numbers of the second kind and the Bernoulli polynomials.

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1. Introduction

We use the relation:

$$S_{n-1}^{[k-1]} = \frac{(-1)^k}{n\,(k-1)!} \sum_{j=k}^{n} \frac{(-1)^j\, j!}{j-k+1}\, S_n^{[j]}, \qquad n \geq 1, \quad 1 \leq k \leq n, \tag{1}$$

where $S_m^{[r]}$ represents a Stirling number of the second kind [1-4], to prove the Roman's identity [5, 6]:

$$S_{n-1}^{[k-1]} = \frac{(-1)^k}{n\,(k-1)!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} B_n(j), \qquad n \geq 1, \quad k = 1, \dots, n, \tag{2}$$

with the participation of the Bernoulli polynomials $B_n(x)$ [3, 7], and the corresponding Bernoulli numbers are given by $B_n = B_n(0)$.

2. Roman's expression

First we shall show that (1) implies the formula:

$$S_{n-1}^{[k-1]} = \frac{k}{n} \sum_{r=0}^{n} \binom{n}{r} B_r\, S_{n-r}^{[k]} \equiv A, \qquad n \geq 1, \quad 1 \leq k \leq n, \tag{3}$$

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in fact, we can employ the relation [8]:

$$B_r = \sum_{q=0}^{r} \frac{(-1)^q\, q!}{q+1}\, S_r^{[q]}, \qquad r = 0, 1, \ldots \tag{4}$$

to obtain:

$$A = \frac{k}{n} \sum_{q=0}^{n} \frac{(-1)^q\, q!}{q+1} \sum_{r=0}^{n} \binom{n}{r} S_r^{[q]}\, S_{n-r}^{[k]}\,,$$

$$= \frac{k}{n} \sum_{q=0}^{n-k} \frac{(-1)^q\, q!}{q+1} \binom{q+k}{q} S_n^{[q+k]}$$

by the convolution formula for Stirling numbers [3],

$$= \frac{(-1)^k}{n\,(k-1)!} \sum_{j=k}^{n} \frac{(-1)^j j!}{j-k+1} S_n^{[j]} \overset{(1)}{=} S_{n-1,}^{[k-1]},$$

hence (3) is correct.

Now in (3) we apply the Euler's expression [1, 3]:

$$S_{n-r}^{[k]} = \frac{(-1)^k}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^{n-r}, \tag{5}$$

then:

$$S_{n-1}^{[k-1]} = \frac{(-1)^k}{n\,(k-1)!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \sum_{r=0}^{n} \binom{n}{r} B_r\, j^{n-r}, \tag{6}$$

but we have the known relationship between the numbers and polynomials of Bernoulli:

$$B_m(x) = \sum_{r=0}^{m} \binom{m}{r} B_r\, j^{m-r}, \tag{7}$$

therefore (6) implies the Roman's identity (2).

3. Bernoulli numbers

We may write (2) in the form:

$$S_n^{[k]} = \frac{1}{(n+1)\,k!} \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+j+1} B_{n+1}(j),$$

then from (4):

$$-(n+1)B_n = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^j B_{n+1}(j) = \sum_{l=1}^{n+1} \frac{1}{l} \left[\sum_{j=1}^{l} (-1)^j \binom{l}{j} B_{n+1}(j) + B_{n+1} \right],$$

$$= \sum_{j=1}^{n+1} (-1)^j B_{n+1}(j) \sum_{l=j}^{n+1} \frac{1}{l} \binom{l}{j} + H_{n+1} \cdot B_{n+1}, \tag{8}$$

where $H_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}$ is a harmonic number [1, 4, 9], besides [10]:

$$\sum_{l=j}^{m} \frac{1}{l} \binom{l}{j} = \frac{1}{j} \binom{m}{j}, \qquad 1 \le j \le m, \tag{9}$$

therefore (8) gives the relation:

$$H_n \cdot B_n + n\, B_{n-1} = -\sum_{j=1}^{n} \frac{(-1)^j}{j} \binom{n}{j} B_n(j), \qquad n \ge 1, \tag{10}$$

that is:

$$\sum_{k=1}^{n} \frac{1}{k} \binom{n}{k} (-1)^k B_n(k) = \begin{cases} -n\, B_{n-1}, & n = 3, 5, 7, \dots \\ -H_n \cdot B_n, & n = 4, 6, 8, \dots \end{cases} \tag{11}$$

here let's remember that $B_0 = 1$, $B_1 = -\frac{1}{2}$ and $B_2 = \frac{1}{6}$.

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