Article

Spin Coefficients & Lanczos Spinor

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Abstract

The covariant derivative $\nabla_{A\dot{B}}$ of the spin-frame metric ε_{CD} is zero, then here we show that this fact implies the existence of a spinor $\gamma_{ABC\dot{D}}$ whose components in the dyad (o^A, ι^B) are the spin coefficients of the corresponding null tetrad. Besides, we exhibit the tensor associated to $\gamma_{EFG\dot{H}}$ via the Infeld-van der Waerden symbols (with the important participation of the Wünsch's vectors), which generates a possible link between the Lanczos spinor and $\gamma_{ABC\dot{D}}$.

Keywords: Newman-Penrose formalism, spin coefficients, Lanczos potential, Wünsch's vectors, null tetrad, two-component spinors.

1. Introduction

The spinor dyad (o^A, ι^B), with $o_A\, \iota^A = 1$, gives the null tetrad of Newman-Penrose [1, 2]:

$$l^\mu \;\leftrightarrow\; o^A\, o^{\dot{B}} \,, \qquad n^\mu \;\leftrightarrow\; \iota^A\, \iota^{\dot{B}} \,, \qquad m^\mu \;\leftrightarrow\; o^A\, \iota^{\dot{B}} \,, \qquad \overline{m}^\mu \;\leftrightarrow\; \iota^A\, o^{\dot{B}} \,, \tag{1}$$

and the spin-frame metric [3]:

$$\varepsilon_{AB} = o_A\, \iota_B - o_B\, \iota_A \,, \tag{2}$$

such that $\nabla_{C\dot{D}}\, \varepsilon_{AB} = 0$, that is:

$$\iota_A\, \nabla_{C\dot{D}}\, o_B - o_A\, \nabla_{C\dot{D}}\, \iota_B = \iota_B\, \nabla_{C\dot{D}}\, o_A - o_B\, \nabla_{C\dot{D}}\, \iota_A \,, \tag{3}$$

which introduces the spinor [4]:

$$\gamma_{EFG\dot{H}} = \iota_E\, \nabla_{C\dot{D}}\, o_F - o_E\, \nabla_{C\dot{D}}\, \iota_F \,, \tag{4}$$

and (3) implies the symmetry $\gamma_{ABC\dot{D}} = \gamma_{BAC\dot{D}}$.

In Sec. 2 we see that the components of (4) in the dyad (o^A, ι^B) are the twelve spin coefficients [1, 2, 5] of the null tetrad (1). In Sec. 3 we employ the Infeld-van der Waerden symbols [6] to determine the tensor associated to $\gamma_{EFG\dot{H}}$, with the important presence of the Wünsch's vectors [7-9], which suggests a possible relationship [4] between (4) and the Lanczos spinor [10, 11].

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2. Spin coefficients

From the analysis realized in [5] is immediate the spinor covariant derivative of the dyad (o^A, ι^B):

$$\nabla_{C\dot{D}}\, o_B = (\gamma\, o_C\, o_{\dot{D}} + \varepsilon\, \iota_C\, \iota_{\dot{D}} - \alpha\, o_C\, \iota_{\dot{D}} - \beta\, \iota_C\, o_{\dot{D}})o_B + (-\tau\, o_C\, o_{\dot{D}} - \kappa\, \iota_C\, \iota_{\dot{D}} + \rho\, o_C\, \iota_{\dot{D}} + \sigma\, \iota_C\, o_{\dot{D}})\iota_B,$$

$$\nabla_{C\dot{D}}\, \iota_B = (\nu\, o_C\, o_{\dot{D}} + \pi\, \iota_C\, \iota_{\dot{D}} - \lambda\, o_C\, \iota_{\dot{D}} - \mu\, \iota_C\, o_{\dot{D}})o_B - (\gamma\, o_C\, o_{\dot{D}} + \varepsilon\, \iota_C\, \iota_{\dot{D}} - \alpha\, o_C\, \iota_{\dot{D}} - \beta\, \iota_C\, o_{\dot{D}})\iota_B, \tag{5}$$

then (4) acquires the structure:

$$\gamma_{ABC\dot{D}} = \iota_A\, \iota_B(\rho\, o_C\, \iota_{\dot{D}} + \sigma\, \iota_C\, o_{\dot{D}} - \tau\, o_C\, o_{\dot{D}} - \kappa\, \iota_C\, \iota_{\dot{D}}) - o_A\, o_B(\nu\, o_C\, o_{\dot{D}} + \pi\, \iota_C\, \iota_{\dot{D}} - \lambda\, o_C\, \iota_{\dot{D}} - \mu\, \iota_C\, o_{\dot{D}}) + (o_A\, \iota_B + o_B\, \iota_A)(\gamma\, o_C\, o_{\dot{D}} + \varepsilon\, \iota_C\, \iota_{\dot{D}} - \alpha\, o_C\, \iota_{\dot{D}} - \beta\, \iota_C\, o_{\dot{D}}), \tag{6}$$

and therefore the spin coefficients of the corresponding null tetrad are given by:

$$\begin{aligned}
&\kappa = o^A\, o^B\, o^C\, o_{\dot{D}}\, \gamma_{ABC}{}^{\dot{D}}, && \sigma = o^A\, o^B\, o^C\, \iota_{\dot{D}}\, \gamma_{ABC}{}^{\dot{D}}, && \rho = o^A\, o^B\, \iota^C\, o_{\dot{D}}\, \gamma_{ABC}{}^{\dot{D}}, \\
&\nu = \iota^A\, \iota^B\, \iota^C\, \iota_{\dot{D}}\, \gamma_{ABC}{}^{\dot{D}}, && \lambda = \iota^A\, \iota^B\, \iota^C\, o_{\dot{D}}\, \gamma_{ABC}{}^{\dot{D}}, && \mu = \iota^A\, \iota^B\, o^C\, \iota_{\dot{D}}\, \gamma_{ABC}{}^{\dot{D}}, \\
&\tau = o^A\, o^B\, \iota^C\, \iota_{\dot{D}}\, \gamma_{ABC}{}^{\dot{D}}, && \varepsilon = \iota^A\, o^B\, o^C o_{\dot{D}}\, \gamma_{ABC}{}^{\dot{D}}, && \alpha = o^A\, \iota^B\, \iota^C\, o_{\dot{D}}\, \gamma_{ABC}{}^{\dot{D}}, \\
&\pi = \iota^A\, \iota^B\, o^C\, o_{\dot{D}}\, \gamma_{ABC}{}^{\dot{D}}, && \gamma = o^A\, \iota^B\, \iota^C\, \iota_{\dot{D}}\, \gamma_{ABC}{}^{\dot{D}}, && \beta = \iota^A o^B o^C \iota_{\dot{D}} \gamma_{ABC}{}^{\dot{D}}.
\end{aligned} \tag{7}$$

The expressions (7) for κ and σ imply that the congruence $\Gamma(l^a)$ is geodesic and shear-free if and only if $o^A\, o^B\, o^C\, \gamma_{ABC\dot{D}} = 0$, that is [12]:

$$\kappa = \sigma = 0 \qquad \Leftrightarrow \qquad o^B\, o^C\, \nabla_{C\dot{D}}\, o_B = 0, \tag{8}$$

which is important to constructing the Kerr-Schild spacetimes [1, 13].

3. Wünsch's vectors

It is easy to see that in (6) are the spinorial versions of the Wünsch's vectors [7-9]:

$$L_a = \nu\, l_a + \pi\, n_a - \lambda\, m_a - \mu\, \overline{m}_a\,, \qquad M_a = \gamma\, l_a + \varepsilon\, n_a - \alpha\, m_a - \beta\, \overline{m}_a\,, \tag{9}$$

$$N_a = -\tau\, l_a - \kappa\, n_a + \rho\, m_a + \sigma\, \overline{m}_a\ ;$$

besides the anti-symmetric tensors [11, 14]:

$$V_{ab} = l_a\, x\, m_b\ , \qquad U_{ab} = \overline{m}_a\, x\, n_b\ , \qquad M_{ab} = m_a\, x\, \overline{m}_b + n_a\, x\, l_b\ , \tag{10}$$

allow to deduce the relations:

$$o_A\, o_B = -\frac{1}{2}\sigma^a{}_A{}^{\dot{Q}}\, \sigma^b{}_{B\dot{Q}}\, V_{ab}\ , \iota_A\, \iota_B = -\frac{1}{2}\sigma^a{}_A{}^{\dot{Q}}\, \sigma^b{}_{B\dot{Q}}\, U_{ab}\ , \quad o_A\, \iota_B + o_B\, \iota_A = \frac{1}{2}\sigma^a{}_A{}^{\dot{Q}}\, \sigma^b{}_{B\dot{Q}}\, M_{ab}\ , \tag{11}$$

therefore if we use (9) and (11) into (6) appears the tensor:

$$W_{abr} \equiv \frac{1}{2}(V_{ab}\, L_r - U_{ab}\, N_r + M_{ab}\, M_r), \tag{12}$$

that under the action of the Infeld-van der Waerden symbols leads to (4):

$$\gamma_{ABC\dot{D}} = \sigma^a{}_A{}^{\dot{Q}}\, \sigma^b{}_{B\dot{Q}}\, \sigma^r{}_{C\dot{D}}\, W_{abr}\ , \tag{13}$$

which is equivalent to (2.18) of Torres del Castillo [6].

In [9] was proved that for arbitrary spacetimes with Petrov types O, N or III [15], the Newman-Penrose components of (9) determine the corresponding Lanczos potential [16-18]; hence the presence in (12) of the Wünsch's vectors suggests some possible link between $\gamma_{ABC\dot{D}}$ and the Lanczos spinor $L_{ABC\dot{D}}$ [8, 10, 11, 19, 20], which we will study in other paper in accordance with the approach indicated by Andersson-Edgar [4].

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