Article

Bianchi Type III Universe Filled with Scalar Field Coupled with Electromagnetic Fields in *f(R, T)* Theory of Gravity

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Abstract

In $f(R,T)$ theory of gravity, we have studied the interacting scalar field and electromagnetic fields in Bianchi type III space-time, by considering the general cases $f(R,T)=f_1(R)+\lambda f_2(T)$, $f(R,T)= f_1(R)f_2(T)$ and $f(R,T)=f(R)$ theory and its particular cases $f(R,T)=R+\lambda T$, $f(R,T)=RT$, $f(R)=R$. It is observed that, even though the cases of $f(R,T)$ theory are distinct, the convergent, non-singular and isotropic solution of metric functions can be evolved in each case along with the components of vector potential, corresponding to suitable integrable function in general cases.

Keywords: Bianchi Type III, *f(R, T)* gravity, scalar field, electromagnetic field.

1. Introduction

Cosmological data from wide range of source have indicated that our universe is undergoing an accelerating expansion [2-8]. To explain this fact, two alternative theories are proposed: one concept of dark energy and other the amendment of general relativity leading to $f(R)$ and $f(R,T)$ theories [3, 4, 5] where R stands for Ricci scalar $R=g^{ij}R_{ij}$, R_{ij} being Ricci tensor T= $g^{ij}T_{ij}$,T_{ij} being energy momentum tensor. The field equations of $f(R,T)$ theories due to Harko [3] are deduced by varying the action

$$s=\int f(R,T)\sqrt{-g}d^4 x+\int L_m\sqrt{-g}d^4x \tag{1.1}$$

where L_m is lagrangian and the other symbols have their usual meaning. Energy momentum tensor is given by

$$T_{ij}=L_m g_{ij}-2\frac{\delta L_m}{\delta g^{ij}} \tag{1.2}$$

Varying the action (1.1) with respect to g^{ij} which yields as

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$$\delta s=\frac{1}{2\chi}\int\left\{f_R(R,T)\frac{\delta R}{\delta g^{ij}}+f_T(R,T)\frac{\delta T}{\delta g^{ij}}+\frac{f(R,T)}{\sqrt{-g}}\frac{\delta(\sqrt{-g})}{\delta g^{ij}}+\frac{2\chi}{\sqrt{-g}}\left(\frac{\delta(L_m\sqrt{-g})}{\delta g^{ij}}\right)\right\}\sqrt{-g}\,d^4x \tag{1.3}$$

Here we define

$$\theta_{ij}=g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{ij}} \tag{1.4}$$

By defining the generalized kronecker symbol $\frac{\delta g^{\alpha\beta}}{\delta g^{ij}}=\delta_i^\alpha\delta_j^\beta$ we can reduce

$$\frac{\delta g^{\alpha\beta}}{\delta g^{ij}}T_{\alpha\beta}=\delta_i^\alpha\delta_j^\beta T_{\alpha\beta}=g^{p\alpha}g_{pi}g^{q\beta}g_{qj}T_{\alpha\beta}=T_{ij}$$

Using above equations we can write

$$\frac{\delta T}{\delta g^{ij}}=\frac{\delta(g^{\alpha\beta}T_{\alpha\beta})}{\delta g^{ij}}=\frac{\delta g^{\alpha\beta}}{\delta g^{ij}}T_{\alpha\beta}+g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{ij}}=T_{ij}+\theta_{ij}$$

Integrating (1.3), we can obtain

$$f_R(R,T)R_{ij}-\frac{1}{2}f(R,T)g_{ij}+\left(g_{ij}\Box-\nabla_i\nabla_j\right)f_R(R,T)=\chi T_j-f_T(R,T)\left[T_{ij}+\theta_{ij}\right] \tag{1.5}$$

This can be further written as

$$f_R(R,T)G_{ij}+\frac{1}{2}[f_R(R,T)R-f(R,T)]g_{ij}+g_{ij}\Box f_R(R,T)-\nabla_i\nabla_j f_R(R,T)$$
$$=\chi T_{ij}-f_T(R,T)\left[T_{ij}+\theta_{ij}\right] \tag{1.6}$$

where $G_{ij}=R_{ij}-\frac{1}{2}Rg_{ij}$

Taking trace of (1.5), we obtain

$$\Box f_R(R,T)=\frac{2}{3}f(R,T)-\frac{1}{3}f_R(R,T)R+\frac{\chi}{3}T-\frac{1}{3}f_T(R,T)[T+\theta] \tag{1.7}$$

Inserting (1.7) in (1.6), we can reorganized as

$$G_j^\mu=\frac{1}{f_R(R,T)}\left[g^{i\mu}\nabla_i\nabla_j f_R(R,T)\right]-\frac{1}{6f_R(R,T)}[f_R(R,T)R+f(R,T)]g_j^\mu$$
$$+\frac{\chi}{f_R(R,T)}\left[T_j^\mu-\frac{1}{3}Tg_j^\mu\right]+\frac{1}{3}\frac{f_T(R,T)}{f_R(R,T)}[T+\theta]g_j^\mu-\frac{f_T(R,T)}{f_R(R,T)}\left[T_j^\mu+\theta_j^\mu\right] \tag{1.8}$$

Let us now calculate the tensor θ_{ij}. Varying (1.2) with respect to metric tensor g^{ij} and using the definition (1.4) we obtain

$$\theta_{ij} = -T_{ij} + 2\left[\frac{\delta L_m}{\delta g^{ij}} - g^{\alpha\beta}\frac{\delta^2 L_m}{\delta g^{ij}\delta g^{\alpha\beta}}\right] \tag{1.9}$$

With this background, in this paper we discover the Bianchi type III space-time with interacting scalar field with electromagnetic one.

2. Matter field Lagrangian L_m

The electromagnetic field tensor

$$F_{ij} = \frac{\partial V_i}{\partial x^j} - \frac{\partial V_j}{\partial x^i},$$

where V_i is electromagnetic four potential.

The aforesaid the matter Lagrangian can be expressed as

$$L_m = \left[\frac{1}{4}F_{\eta\tau}F^{\eta\tau} - \frac{1}{2}\varphi_{,\eta}\varphi^{,\eta}\psi\right], \tag{2.1}$$

where $\psi = \psi(I),\ I = V_i V^i$

The functionψ characterizes the interaction between the scalarφ and electromagnetic field [8].

Then the energy momentum tensor in (1.2) can conveniently be expressed in the mixed form

$$T_j^i = \left(F_\alpha^i F_j^\alpha + \frac{1}{4}g_j^i F_{\alpha\beta}F^{\alpha\beta}\right) - \left[\frac{1}{2}\psi g_j^i - \dot{\psi}V^i V_j\right]\varphi_{,\eta}\varphi^{,\eta} + \psi\varphi^{,i}\varphi_{,j} \tag{2.2}$$

Similarly the tensor θ_j^i in (1.9) can be written in mixed form as

$$\theta_j^i = -T_j^i - \left(\psi - I\dot{\psi}\right)\varphi^{,i}\varphi_{,j} + I\ddot{\psi}\varphi_{,\eta}\varphi^{,\eta}V^i V_j \tag{2.3}$$

The equations (2.2) and (2.3), after contraction yield

$$T = -(\psi - I\dot{\psi})\varphi_{,\eta}\varphi^{,\eta} \tag{2.4}$$

$$\theta = I^2\ddot{\psi}\varphi_{,\eta}\varphi^{,\eta} \tag{2.5}$$

3. Bianchi type III space-time

We consider the Bianchi type III space-time specified by

$$ds^2 = A^2dx^2 + B^2e^{-2mx}dy^2 + C^2dz^2 - dt^2, \tag{3.1}$$

where A, B, C are functions of t and m is non-zero constant.

The non-vanishing components of Einstein tensor are

$$G_1^1 = \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC}\ , \qquad G_2^2 = \frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC}$$

$$G_3^3 = -\frac{m^2}{A^2} + \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB}\ , \qquad G_4^1 = \frac{m}{A^2}\left[\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right]$$

Electromagnetic field tensor F_{ij}

To achieve the compatibility with the non-static space time (3.1), we assume the electromagnetic vector potential in the form

$$V_i = [\alpha(x)V_1(t), V_2(t), V_3(t), V_4(t)]\ , \tag{3.2}$$

Then it is easy to deduce

$$I = \left[\frac{\alpha^2 {V_1}^2}{A^2} + \frac{{V_2}^2}{B^2} e^{2mx} + \frac{{V_3}^2}{C^2} - {V_4}^2\right] \tag{3.3}$$

$$F_{14} = \alpha \dot{V}_1\ , \quad F_{24} = \dot{V}_2\ , \quad F_{34} = \dot{V}_3 \tag{3.4}$$

$$F_{ij}F^{ij} = -2\left[\frac{\alpha^2 {\dot{V}_1}^2}{A^2} + \frac{{\dot{V}_2}^2}{B^2} e^{2-x} + \frac{{\dot{V}_3}^2}{C^2}\right] \tag{3.5}$$

$$\varphi_i \varphi^i = -\dot{\varphi}^2 \tag{3.6}$$

In reference to the above quantities at our disposal, the components of energy momentum tensors from (2.2) becomes

$$T_1^1 = \frac{1}{2}\frac{\alpha^2 {\dot{V}_1}^2}{A^2} - \frac{1}{2}\frac{{\dot{V}_2}^2}{B^2} e^{2mx} - \frac{1}{2}\frac{{\dot{V}_3}^2}{C^2} + \frac{1}{2}\psi\dot{\varphi}^2 - \dot{\psi}\dot{\varphi}^2 \frac{\alpha^2 {V_1}^2}{A^2} \tag{3.7a}$$

$$T_2^1 = \frac{\alpha \dot{V}_1 \dot{v}_2}{A^2} - \dot{\psi}\dot{\varphi}^2 \frac{\alpha V_1 V_2}{A^2} \tag{3.7b}$$

$$T_3^1 = \frac{\alpha \dot{V}_1 \dot{v}_3}{A^2} - \dot{\psi}\dot{\varphi}^2 \frac{\alpha V_1 V_3}{A^2} \tag{3.7c}$$

$$T_4^1 = \dot{\psi}\dot{\varphi}^2 \frac{\alpha V_1 V_4}{A^2} \tag{3.7d}$$

$$T_2^2 = -\frac{1}{2}\frac{\alpha^2 {\dot{V}_1}^2}{A^2} + \frac{1}{2}\frac{{\dot{V}_2}^2}{B^2} e^{2mx} - \frac{1}{2}\frac{{\dot{V}_3}^2}{C^2} + \frac{1}{2}\psi\dot{\varphi}^2 - \dot{\psi}\dot{\varphi}^2 \frac{{V_2}^2}{B^2} \tag{3.7e}$$

$$T_3^2 = \frac{\dot{V}_2\dot{v}_3}{B^2}e^{2mx} - \dot{\psi}\dot{\varphi}^2\frac{V_2V_3}{B^2}e^{2mx} \quad (3.7f)$$

$$T_3^3 = -\frac{1}{2}\frac{\alpha^2\dot{V_1}^2}{A^2} - \frac{1}{2}\frac{\dot{V_2}^2}{B^2}e^{2mx} + \frac{1}{2}\frac{\dot{V_3}^2}{C^2} + \frac{1}{2}\psi\dot{\varphi}^2 - \dot{\psi}\dot{\varphi}^2\frac{{V_3}^2}{C^2} \quad (3.7g)$$

$$T_4^4 = \frac{1}{2}\frac{\alpha^2\dot{V_1}^2}{A^2} + \frac{1}{2}\frac{\dot{V_2}^2}{B^2}e^{2mx} + \frac{1}{2}\frac{\dot{V_3}^2}{C^2} - \frac{1}{2}\psi\dot{\varphi}^2 + \dot{\psi}\dot{\varphi}^2{V_4}^2 \quad (3.7h)$$

$$T = (\psi - I\dot{\psi})\dot{\varphi}^2 \quad (3.7i)$$

Similarly the components of tensors θ_j^i from (2.3), assumes the values

$$\theta_1^1 = -T_1^1 - I\ddot{\psi}\dot{\varphi}^2\frac{\alpha^2{V_1}^2}{A^2} \quad (3.8a)$$

$$\theta_2^1 = -T_2^1 - I\ddot{\psi}\dot{\varphi}^2\frac{\alpha V_1V_2}{A^2} \quad (3.8b)$$

$$\theta_3^1 = -T_3^1 - I\ddot{\psi}\dot{\varphi}^2\frac{\alpha V_1V_3}{A^2} \quad (3.8c)$$

$$\theta_4^1 = -T_4^1 - I\ddot{\psi}\dot{\varphi}^2\frac{\alpha V_1V_4}{A^2} \quad (3.8d)$$

$$\theta_2^2 = -T_2^2 - I\ddot{\psi}\dot{\varphi}^2\frac{{V_2}^2}{B^2}e^{2mx} \quad (3.8e)$$

$$\theta_3^2 = -T_3^2 - I\ddot{\psi}\dot{\varphi}^2\frac{V_2V_3}{B^2}e^{2mx} \quad (3.8f)$$

$$\theta_3^3 = -T_3^3 - I\ddot{\psi}\dot{\varphi}^2\frac{{V_3}^2}{C^2} \quad (3.8g)$$

$$\theta_4^4 = -T_4^4 + \left(\psi - I\dot{\psi}\right)\dot{\varphi}^2 + I\ddot{\psi}\dot{\varphi}^2{V_4}^2 \quad (3.8h)$$

$$\theta = -I^2\ddot{\psi}\dot{\varphi}^2 \quad (3.8i)$$

Following Saha [1] the variation of the matter LagrangiamL_m in (2.1) with respect to the electromagnetic field gives us

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^j}\left(\sqrt{-g}F^{ij}\right) - \left(\varphi_{,j}\varphi^{,j}\right)\dot{\psi}A^i = 0\,, \quad \text{where} \quad \dot{\psi} = \frac{\partial\psi}{\partial I}$$

Noting (3.2) and (3.4) above equation gives the following

$$\text{for } i = 1, j = 4 \Rightarrow \left(\frac{\dot{V}_1}{V_1}\right)^{\cdot} + \frac{\dot{V_1}^2}{{V_1}^2} + \frac{\dot{V}_1}{V_1}\left[\frac{\dot{C}}{C} + \frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right] = \dot{\psi}\dot{\varphi}^2 \quad (3.9a)$$

$$\text{for } i = 2, j = 4 \Rightarrow \left(\frac{\dot{V}_2}{V_2}\right)^{\cdot} + \frac{\dot{V_2}^2}{{V_2}^2} + \frac{\dot{V}_2}{V_2}\left[\frac{\dot{A}}{A} + \frac{\dot{C}}{C} - \frac{\dot{B}}{B}\right] = \dot{\psi}\dot{\varphi}^2 \quad (3.9b)$$

for $i = 3, j = 4 \Rightarrow \left(\frac{\dot{V}_3}{V_3}\right)^{\cdot} + \frac{\dot{V}_3{}^2}{V_3{}^2} + \frac{\dot{V}_3}{V_3}\left[\frac{\dot{B}}{B} + \frac{\dot{A}}{A} - \frac{\dot{C}}{C}\right] = \dot{\psi}\dot{\varphi}^2$ (3.9c)

for $i = 4, j = 1 \Rightarrow \alpha(x) = k_1 e^{mx}$ (3.9d)

for $i = 4, j = 4 \Rightarrow V_4 = 0$, (3.9e)

where k_1 is constant of integration.

Since the expression of the Einstein tensor in (1.8) is complicated, the solution of the Einstein field equation in general cannot be obtained. With this reality we take recourse to the particular cases of the function $f(R,T)$ and there upon try to obtain the solution.

4. Sub case $f(R,T) = f_1(R) + \lambda f_2(T)$

In this case we follow the notations

$$f_R(R,T) = \frac{\partial f(R,T)}{\partial R} = \dot{f}_1(R) , \quad f_T(R,T) = \frac{\partial f(R,T)}{\partial T} = \lambda \dot{f}_2(T)$$

Then (1.8) reduces to the form

$$G_j^{\mu} = \frac{1}{\dot{f}_1(R)}\left[g^{i\mu}\nabla_i\nabla_j \dot{f}_1(R)\right] - \frac{1}{6\dot{f}_1(R)}\left[\dot{f}_1(R)R + f_1(R) + \lambda f_2(T)\right]g_j^{\mu} + \frac{\chi}{\dot{f}_1(R)}\left[T_j^{\mu} - \frac{1}{3}Tg_j^{\mu}\right] + \frac{\lambda}{3}\frac{\dot{f}_2(T)}{\dot{f}_1(R)}[T+\theta]g_j^{\mu} - \frac{\lambda \dot{f}_2(T)}{\dot{f}_1(R)}\left[T_j^{\mu} + \theta_j^{\mu}\right] \quad (4.1)$$

Since for the space time (3.1), we have

$$G_2^1 = 0, \; G_3^1 = 0, \; G_3^2 = 0$$

Using (3.7) and (3.8), the field equations (4.1) yield

$$\frac{\dot{V}_1\dot{V}_2}{V_1V_2} = \dot{\psi}\dot{\varphi}^2 - \frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2 \quad (4.2a)$$

$$\frac{\dot{V}_1\dot{V}_3}{V_1V_3} = \dot{\psi}\dot{\varphi}^2 - \frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2 \quad (4.2b)$$

$$\frac{\dot{V}_2\dot{V}_3}{V_2V_3} = \dot{\psi}\dot{\varphi}^2 - \frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2 \quad (4.2c)$$

From (4.2), we can write

$$\frac{\dot{V}_1\dot{V}_2}{V_1V_2} = \frac{\dot{V}_2\dot{V}_3}{V_2V_3} = \frac{\dot{V}_1\dot{V}_3}{V_1V_3} = \dot{\psi}\dot{\varphi}^2 - \frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2 \quad (4.3)$$

or we can rewrite it as

$$\frac{\dot{V}_1}{V_1} = \frac{\dot{V}_2}{V_2} = \frac{\dot{V}_3}{V_3} = \frac{\dot{h}_1}{h_1} \text{ say ,} \tag{4.4}$$

where h_1 is some unknown function of t.

Inserting (4.4) in (4.3) it yields

$$\left(\frac{\dot{h}_1}{h_1}\right)^2 = \left(\frac{\dot{h}_1}{h_1}\right)^2 = \left(\frac{\dot{h}_1}{h_1}\right)^2 = \dot{\psi}\dot{\varphi}^2 - \frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2 \tag{4.5}$$

Up on the integration of equation (4.4) with respect to t, we get

$$V_1 = k_2 h_1, \qquad V_2 = k_3 h_1\ , \quad V_3 = k_4 h_1 \tag{4.6}$$

where k_2, k_3, k_4 are constants of integration.

Now our plan is to express the components of T_j^i in (3.7) in terms of T_4^4. For this we consider the expression

$$\frac{\alpha^2 \dot{V_1}^2}{A^2} + \frac{\dot{V_2}^2}{B^2}e^{2mx} + \frac{\dot{V_3}^2}{C^2} = \left[\frac{\alpha^2 {V_1}^2}{A^2} + \frac{{V_2}^2}{B^2}e^{2mx} + \frac{{V_3}^2}{C^2}\right]\left(\frac{\dot{h}_1}{h_1}\right)^2 \quad \text{by (4.4)}$$

$$= I\left(\frac{\dot{h}_1}{h_1}\right)^2 \qquad \text{by (3.3) and (3.9e)}$$

$$= I\dot{\psi}\dot{\varphi}^2 - \frac{\lambda}{\chi}\dot{f}_2(T)I^2\ddot{\psi}\dot{\varphi}^2 \quad \text{by (4.5)} \tag{4.7}$$

We attempt to express the components of T_j^i in (3.7) in terms of T_4^4 by using (4.4), (4.5) and (4.7) as follows

$$T_4^4 = \frac{1}{2}I\dot{\psi}\dot{\varphi}^2 - \frac{1}{2}\frac{\lambda}{\chi}\dot{f}_2(T)I^2\ddot{\psi}\dot{\varphi}^2 - \frac{1}{2}\psi\dot{\varphi}^2 \tag{4.8a}$$

$$T_1^1 = -T_4^4 - \frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2\frac{\alpha^2 {V_1}^2}{A^2} \tag{4.8b}$$

$$T_2^1 = -\frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2\frac{\alpha V_1 v_2}{A^2} \tag{4.8c}$$

$$T_3^1 = -\frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2\frac{\alpha V_1 v_3}{A^2} \tag{4.8d}$$

$$T_4^1 = 0 \tag{4.8e}$$

$$T_2^2 = -T_4^4 - \frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2\frac{{V_2}^2}{B^2}e^{2m} \tag{4.8f}$$

$$T_3^2 = -\frac{\lambda}{\chi}\dot{f}_2(T) I\ddot{\psi}\dot{\varphi}^2 \frac{V_2 v_3}{B^2} \tag{4.8g}$$

$$T_3^3 = -T_4^4 - \frac{\lambda}{\chi}\dot{f}_2(T) I\ddot{\psi}\dot{\varphi}^2 \frac{{V_3}^2}{C^2} \tag{4.8h}$$

$$T = (\psi - I\dot{\psi})\dot{\varphi}^2 \tag{4.8i}$$

We consider the non-vanishing components of Einstein tensors $G_1^1, G_2^2, G_3^3, G_4^1$ from (4.1)

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = \frac{\dot{A}}{A}\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt} - \frac{1}{6\dot{f}_1(R)}\left[\dot{f}_1(R)R + f_1(R) + \lambda f_2(T)\right] + \frac{\chi}{\dot{f}_1(R)}\left[T_1^1 - \frac{1}{3}T\right] + \frac{\lambda\dot{f}_2(T)}{3\dot{f}_1(R)}[T + \theta] - \frac{\lambda\dot{f}_2(T)}{3\dot{f}_1(R)}[T_1^1 + \theta_1^1] \tag{4.9a}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} = \frac{\dot{B}}{B}\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt} - \frac{1}{6\dot{f}_1(R)}\left[\dot{f}_1(R)R + f_1(R) + \lambda f_2(T)\right] + \frac{\chi}{\dot{f}_1(R)}\left[T_2^2 - \frac{1}{3}T\right] + \frac{\lambda\dot{f}_2(T)}{3\dot{f}_1(R)}[T + \theta] - \frac{\lambda\dot{f}_2(T)}{3\dot{f}_1(R)}[T_2^2 + \theta_2^2] \tag{4.9b}$$

$$-\frac{m^2}{A^2} + \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{Ab} = \frac{\dot{C}}{C}\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dD} - \frac{1}{6\dot{f}_1(R)}\left[\dot{f}_1(R)R + f_1(R) + \lambda f_2(T)\right] + \frac{\chi}{\dot{f}_1(R)}\left[T_3^3 - \frac{1}{3}T\right] + \frac{\lambda\dot{f}_2(T)}{3\dot{f}_1(R)}[T + \theta] - \frac{\lambda\dot{f}_2(T)}{3\dot{f}_1(R)}[T_3^3 + \theta_3^3] \tag{4.9c}$$

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = 0 \tag{4.9d}$$

Upon integration of the equation (4.9d), we obtain

$$A = k_5 B \tag{4.9e}$$

where k_5 is constant of integration.

Subtracting (4.9b) from (4.9a), (4.9c) from (4.9b) and (4.9a) from (4.9c) we get

$$\frac{\ddot{B}}{B} - \frac{\ddot{A}}{A} + \frac{\dot{C}}{C}\left[\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right] + \left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right)\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt} = \frac{\chi}{\dot{f}_1(R)}[T_1^1 - T_2^2] + \frac{\lambda\dot{f}_2(T)}{\dot{f}_1(R)}[(T_2^2 + \theta_2^2) - (T_1^1 + \theta_1^1)] \tag{4.10a}$$

$$\frac{\ddot{C}}{C} - \frac{\ddot{B}}{B} + \frac{\dot{A}}{A}\left[\frac{\dot{-}}{C} - \frac{\dot{B}}{B}\right] + \left(\frac{\dot{C}}{C} - \frac{\dot{B}}{B}\right)\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt} + \frac{m^2}{A^2} = \frac{\chi}{\dot{f}_1(R)}[T_2^2 - T_3^3] + \frac{\lambda\dot{f}_2(T)}{\dot{f}_1(R)}[(T_3^3 + \theta_3^3) - (T_2^2 + \theta_2^2)] \tag{4.10b}$$

$$\frac{\ddot{A}}{A} - \frac{\ddot{C}}{c} + \frac{\dot{B}}{B}\left[\frac{\dot{A}}{A} - \frac{\dot{C}}{C}\right] + \left(\frac{\dot{A}}{A} - \frac{\dot{C}}{C}\right)\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt} - \frac{m^2}{A^2} = \frac{\chi}{\dot{f}_1(R)}[T_3^3 - T_1^1] + \frac{\lambda\dot{f}_2(T)}{\dot{f}_1(R)}[(T_1^1 + \theta_1^1) - (T_3^3 + \theta_3^3)] \tag{4.10c}$$

Using (3.8) and (4.8) the equations (4.10) reduces

$$\frac{\ddot{B}}{B}-\frac{\ddot{A}}{A}+\frac{\dot{C}}{C}\left[\frac{\dot{B}}{B}-\frac{\dot{A}}{A}\right]+\left(\frac{\dot{B}}{B}-\frac{\dot{A}}{A}\right)\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}=0 \quad (4.11a)$$

$$\frac{\ddot{C}}{C}-\frac{\ddot{B}}{B}+\frac{\dot{A}}{A}\left[\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\right]+\left(\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\right)\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}+\frac{m^2}{A^2}=0 \quad (4.11b)$$

$$\frac{\ddot{A}}{A}-\frac{\ddot{C}}{c}+\frac{\dot{B}}{B}\left[\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right]+\left(\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right)\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}-\frac{m^2}{A^2}=0 \quad (4.11c)$$

Eliminating $\frac{m^2}{A^2}$ between the equations (4.11b) and (4.11c), we obtain

$$\frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}+\frac{\dot{C}}{C}\left[\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right]+\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}=0 \quad (4.11d)$$

Upon integration of the equations (4.11a) and (4.11d) yield

$$\frac{A}{B}=k_7 exp\left\{k_6\int\frac{1}{ABC\dot{f}_1(R)}dt\right\} \quad (4.12a)$$

$$\frac{B}{A}=k_9\exp\left\{k_8\int\frac{1}{ABC\dot{f}_1(R)}dt\right\} \quad (4.12b)$$

where $k's$ are constants of integration, such that

$$k_7k_9=1 \text{ and } k_6+k_8=0$$

By using (4.4) we can write the equation (3.9) as

$$\left(\frac{\dot{h}_1}{h_1}\right)^{\cdot}+\frac{\dot{h}_1{}^2}{h_1{}^2}+\frac{\dot{h}_1}{h_1}\left[\frac{\dot{C}}{C}+\frac{\dot{B}}{B}-\frac{\dot{A}}{A}\right]=\dot{\psi}\dot{\varphi}^2 \quad (4.13a)$$

$$\left(\frac{\dot{h}_1}{h_1}\right)^{\cdot}+\frac{\dot{h}_1{}^2}{h_1{}^2}+\frac{\dot{h}_1}{h_1}\left[\frac{\dot{A}}{A}+\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\right]=\dot{\psi}\dot{\varphi}^2 \quad (4.13b)$$

$$\left(\frac{\dot{h}_1}{h_1}\right)^{\cdot}+\frac{\dot{h}_1{}^2}{h_1{}^2}+\frac{\dot{h}_1}{h_1}\left[\frac{\dot{B}}{B}+\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right]=\dot{\psi}\dot{\varphi}^2 \quad (4.13c)$$

This equation further imply that

$$\frac{\dot{C}}{C}+\frac{\dot{B}}{B}-\frac{\dot{A}}{A}=\frac{\dot{A}}{A}+\frac{\dot{C}}{C}-\frac{\dot{B}}{B}=\frac{\dot{B}}{B}+\frac{\dot{A}}{A}-\frac{\dot{C}}{C}$$

or $$\frac{\dot{A}}{A}=\frac{\dot{B}}{B}=\frac{\dot{C}}{C} \quad (4.14)$$

Upon integration of the equation (4.14) yields

$$A=k_{10}B,\quad B=k_{11}\,C,\quad C=k_{12}A \quad (4.15)$$

where k_{10}, k_{11}, k_{12} are constants of integration.

We observe that C is scalar multiple of A, therefore we can write explicitly as

$$A = (A^2B)^{\frac{1}{3}} k_{13} exp\left\{k_{14}\int \frac{1}{ABC\dot{f}_1(R)}dt\right\} \tag{4.16a}$$

$$B = (A^2B)^{\frac{1}{3}} k_{15} exp\left\{k_{16}\int \frac{1}{ABC\dot{f}_1(R)}dt\right\} \tag{4.16b}$$

$$C = (A^2B)^{\frac{1}{3}} k_{17} exp\left\{k_{18}\int \frac{1}{ABC\dot{f}_1(R)}dt\right\} \tag{4.16c}$$

If we convert A into C we can rewrite as

$$A = (ABC)^{\frac{1}{3}} k_{9} exp\left\{k_{14}\int \frac{1}{ABC\dot{f}_1(R)}dt\right\} \tag{4.17a}$$

$$B = (ABC)^{\frac{1}{3}} k_{20} exp\left\{k_{16}\int \frac{1}{ABC\dot{f}_1(R)}dt\right\} \tag{4.17b}$$

$$C = (AB)^{\frac{1}{3}} k_{21} exp\left\{k_{18}\int \frac{1}{ABC\dot{f}_1(R)}dt\right\} \tag{4.17c}$$

where $k's$ are constants of integration.

Inserting (4.14) in (4.13)

$$\left(\frac{\dot{h}_1}{h_1}\right)^{\cdot} + \frac{\dot{h}_1{}^2}{h_1{}^2} + \frac{\dot{h}_1}{h_1}\left[\frac{\dot{A}}{A}\right] = \dot{\psi}\dot{\varphi}^2 \tag{4.18}$$

But from (4.5) we have

$$\dot{\psi}\dot{\varphi}^2 = \left(\frac{\dot{h}_1}{h_1}\right)^2 + \frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2 \tag{4.19}$$

Inserting (4.19) in (4.18) we have

$$\left(\frac{\dot{h}_1}{h_1}\right)^{\cdot} + \frac{\dot{h}_1}{h_1}\left[\frac{\dot{A}}{A}\right] = \frac{\lambda}{\chi}\dot{f}_2(T)I\ddot{\psi}\dot{\varphi}^2 \tag{4.20}$$

If we confine the function ψ as linear function $\ddot{\psi} = 0$ or $\psi = k_{22}I + k_{23}$ then the equation (4.20) has perfect solution

$$h_1 = k_{25} exp\left\{k_{24}\int \frac{1}{A}dt\right\} \tag{4.21}$$

With the help of (4.21), the equations (4.6) convert in to

$$V_1 = k_{26} exp\left\{k_{24}\int \frac{1}{A}dt\right\} \tag{4.22a}$$

$$V_2 = k_{27} exp\left\{k_{24} \int \frac{1}{A} dt\right\} \tag{4.22b}$$

$$V_3 = k_{28} exp\left\{k_{24} \int \frac{1}{A} dt\right\} \tag{4.22c}$$

where $k's$ are constant of integration.

5. Subcase $f(R,T) = f_1(R) f_2(T)$

In this case we follow the notations

$$f_R(R,T) = \frac{\partial f(R,T)}{\partial R} = \dot{f}_1(R) f_2(T)\ ,\ f_T(R,T) = \frac{\partial f(R,T)}{\partial T} = f_1(R)\dot{f}_2(T) \tag{5.1}$$

With the help of (5.1), the field equation (1.8) reduces to

$$G_j^i = \frac{1}{\dot{f}_1(R) f_2(T)}\left[g^{im}\nabla_m\nabla_j \dot{f}_1(R) f_2(T)\right] - \frac{1}{6\dot{f}_1(R) f_2(T)}\left[\dot{f}_1(R) f_2(T)\mathrm{R} + f_1(R) f_2(T)\right] g_j^i$$

$$+ \frac{\chi}{f_1(R) f_2(T)}\left[T_j^i - \frac{1}{3} T g_j^i\right] + \frac{1}{3}\frac{f_1(R)\dot{f}_2(T)}{\dot{f}_1(R) f_2(T)}\left[T+\theta\right] g_j^i - \frac{f_1(R)\dot{f}_2(T)}{\dot{f}_1(R) f_2(T)}\left[T_j^i + \theta_j^i\right] \tag{5.2}$$

Since for the space-time (3.1), we have

$$G_2^1 = 0,\ G_3^1 = 0,\ G_3^2 = 0$$

Using (3.7) and (3.8), the field equation (5.2) yield

$$\frac{\dot{V}_1 \dot{V}_2}{V_1 V_2} = \dot{\psi}\dot{\varphi}^2 - \frac{f_1(R)\dot{f}_2(T)}{\chi} I \ddot{\psi}\dot{\varphi}^2 \tag{5.3a}$$

$$\frac{\dot{V}_1 \dot{V}_3}{V_1 V_3} = \dot{\psi}\dot{\varphi}^2 - \frac{f_1(R)\dot{f}_2(T)}{\chi} I \ddot{\psi}\dot{\varphi}^2 \tag{5.3b}$$

$$\frac{\dot{V}_2 \dot{V}_3}{V_2 V_3} = \dot{\psi}\dot{\varphi}^2 - \frac{f_1(R)\dot{f}_2(T)}{\chi} I \ddot{\psi}\dot{\varphi}^2 \tag{5.3c}$$

From (5.3) we can write

$$\frac{\dot{V}_1 \dot{V}_2}{V_1 V_2} = \frac{\dot{V}_2 \dot{V}_3}{V_2 V_3} = \frac{\dot{V}_1 \dot{V}_3}{V_1 V_3} = \dot{\psi}\dot{\varphi}^2 - \frac{f_1(R)\dot{f}_2(T)}{\chi} I \ddot{\psi}\dot{\varphi}^2 \tag{5.4}$$

or $\frac{\dot{V}_1}{V_1} = \frac{\dot{V}_2}{V_2} = \frac{\dot{V}_3}{V_3} \equiv \frac{\dot{h}_7}{h_7}$, say (5.5)

Inserting (5.5) in (5.4), we get

$$\left(\frac{\dot{h}_7}{h_7}\right)^2 = \left(\frac{\dot{h}_7}{h_7}\right)^2 = \left(\frac{\dot{h}_7}{h_7}\right)^2 = \dot{\psi}\dot{\varphi}^2 - \frac{f_1(R)\dot{f}_2(T)}{\chi} I \ddot{\psi}\dot{\varphi}^2 \tag{5.6}$$

Up on integration of the equation (5.5) yield

$$V_1 = m_1 h_7 V_2 = m_2 h_7 V_3 = m_3 h_7 \tag{5.7}$$

where m_1, m_2, m_3 are constants of integration.

Now our plan is to express the components of T_j^i in (3.7) in terms of T_4^4. For this we consider the expression

$$\frac{\alpha^2 \dot{V_1}^2}{A^2} + \frac{\dot{V_2}^2}{B^2} e^{2mx} + \frac{\dot{V_3}^2}{C^2} = \left[\frac{\alpha^2 {V_1}^2}{A^2} + \frac{{V_2}^2}{B^2} e^{2mx} + \frac{{V_3}^2}{C^2}\right] \left(\frac{\dot{h}_7}{h_7}\right)^2 \text{ by (5.5)}$$

$$= I \left(\frac{\dot{h}_7}{h_7}\right)^2 \text{ by (3.3) and (3.9e)}$$

$$= I\dot{\psi}\dot{\varphi}^2 - \frac{f_1(R)\dot{f}_2(T}{\chi} I^2 \ddot{\psi}\dot{\varphi}^2 \quad \text{by (5.6)} \tag{5.8}$$

We attempt to express the components of T_j^i in (3.7) in terms of T_4^4 by using (5.5), (5.6) and (5.8) as follows

$$T_4^4 = \frac{1}{2} I\dot{\psi}\dot{\varphi}^2 - \frac{1}{2}\frac{f_1(R)\dot{f}_2(T}{\chi} I^2 \ddot{\psi}\dot{\varphi}^2 - \frac{1}{2}\psi\dot{\varphi}^2 \tag{5.9a}$$

$$T_1^1 = -T_4^4 - \frac{f_1(R)\dot{f}_2(T)}{\chi} I\ddot{\psi}\dot{\varphi}^2 \frac{\alpha^2 {V_1}^2}{A^2} \tag{5.9b}$$

$$T_2^1 = -\frac{f_1(R)\dot{f}_2(T)}{\chi} I\ddot{\psi}\dot{\varphi}^2 \frac{\alpha V_1 V_2}{A^2} \tag{5.9c}$$

$$T_3^1 = -\frac{f_1(R)\dot{f}_2(T)}{\chi} I\ddot{\psi}\dot{\varphi}^2 \frac{\alpha V_1 V_3}{A^2} \tag{5.9d}$$

$$T_4^1 = 0 \tag{5.9e}$$

$$T_2^2 = -T_4^4 - \frac{f_1(R)\dot{f}_2(T)}{\chi} I\ddot{\psi}\dot{\varphi}^2 \frac{{V_2}^2}{B^2} e^{2mx} \tag{5.9f}$$

$$T_3^2 = -\frac{f_1(R)\dot{f}_2(T)}{\chi} I\ddot{\psi}\dot{\varphi}^2 \frac{V_2 V_3}{B^2} e^{2mx} \tag{5.9g}$$

$$T_3^3 = -T_4^4 - \frac{f_1(R)\dot{f}_2(T)}{\chi} I\ddot{\psi}\dot{\varphi}^2 \frac{{V_3}^2}{C^2} \tag{5.9h}$$

$$T = (\psi - I\dot{\psi})\dot{\varphi}^2 \tag{5.9i}$$

We consider the non-vanishing components of Einstein tensor G_1^1, G_2^2, G_3^3 from (5.2)

$$\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+\frac{\dot{B}\dot{C}}{BC}=\frac{1}{A^2}\frac{\ddot{f}_2(T)}{f_2(T)}\left(\frac{dT}{dx}\right)^2+\frac{1}{A^2}\frac{\dot{f}_2(T)}{f_2(T)}\frac{d^2T}{dx^2}+\frac{\dot{A}}{A}\left[\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}+\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dt}\right]-\frac{1}{6}\left[R+\frac{f_1(R)}{\dot{f}_1(R)}\right]$$

$$+\frac{\chi}{\dot{f}_1(R)f_2(T)}\left[T_1^1-\frac{1}{3}T\right]+\frac{1}{3}\frac{f_1(R)\dot{f}_2(T)}{\dot{f}_1(R)f_2(T)}[T+\theta]-\frac{f_1(R)\dot{f}_2(T)}{\dot{f}_1(R)f_2(T)}\left[T_1^1+\theta_1^1\right] \quad (5.10a)$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{C}}{C}+\frac{\dot{A}\dot{C}}{AC}=\frac{m}{A^2}\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dx}+\frac{\dot{B}}{B}\left[\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}+\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dt}\right]-\frac{1}{6}\left[R+\frac{f_1(R)}{\dot{f}_1(R)}\right]$$

$$+\frac{\chi}{\dot{f}_1(R)f_2(T)}\left[T_2^2-\frac{1}{3}T\right]+\frac{1}{3}\frac{f_1(R)\dot{f}_2(T)}{\dot{f}_1(R)f_2(T)}[T+\theta]-\frac{f_1(R)\dot{f}_2(T)}{\dot{f}_1(R)f_2(T)}[T_2^2+\theta_2^2] \quad (5.10b)$$

$$-\frac{m^2}{A^2}+\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{B}}{AB}=\frac{\dot{C}}{c}\left[\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}+\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dt}\right]-\frac{1}{6}\left[R+\frac{f_1(R)}{\dot{f}_1(R)}\right]+\frac{\chi}{\dot{f}_1(R)f_2(T)}\left[T_3^3-\frac{1}{3}T\right]$$

$$+\frac{1}{3}\frac{f_1(R)\dot{f}_2(T)}{\dot{f}_1(R)f_2(T)}[T+\theta]-\frac{f_1(R)\dot{f}_2(T)}{\dot{f}_1(R)f_2(T)}[T_3^3+\theta_3^3] \quad (5.10c)$$

Subtracting (5.10b) from (5.10a), (5.10c) from (5.10b) and (5.10a) from (5.10c) we obtain

$$\frac{\ddot{B}}{B}-\frac{\ddot{A}}{A}+\frac{\dot{C}}{C}\left[\frac{\dot{B}}{B}-\frac{\dot{A}}{A}\right]+\left(\frac{\dot{B}}{B}-\frac{\dot{A}}{A}\right)\left[\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}+\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dt}\right]$$

$$=\frac{1}{A^2}\left[\frac{\ddot{f}_2(T)}{f_2(T)}\left(\frac{dT}{dx}\right)^2+\frac{\dot{f}_2(T)}{f_2(T)}\frac{d^2T}{dx^2}-m\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dx}\right]+\frac{\chi}{\dot{f}_1(R)-_2(T)}[T_1^1-T_2^2]$$

$$+\frac{f_1(R)\dot{f}_2(T)}{\dot{f}_1(R)f_2(T)}[(T_2^2+\theta_2^2)-(T_1^1+\theta_1^1)] \quad (5.11a)$$

$$\frac{\ddot{C}}{C}-\frac{\ddot{B}}{B}+\frac{\dot{A}}{A}\left[\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\right]+\left(\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\right)\left[\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}+\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dt}\right]$$

$$=-\frac{m^2}{A^2}+\frac{m}{A^2}\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dx}+\frac{\chi}{\dot{f}_1(R)f_2(T)}[T_2^2-T_3^3]+\frac{f_1(R)\dot{f}_2(T)}{\dot{f}_1(R)f_2(T)}[(T_3^3+\theta_3^3)-(T_2^2+\theta_2^2)] \quad (5.11b)$$

$$\frac{\ddot{A}}{A}-\frac{\ddot{C}}{C}+\frac{\dot{B}}{B}\left[\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right]+\left(\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right)\left[\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}+\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dt}\right]$$

$$=\frac{m^2}{A^2}-\frac{1}{A^2}\frac{\ddot{f}_2(T)}{f_2(T)}\left(\frac{dT}{dx}\right)^2-\frac{1}{A^2}\frac{\dot{f}_2(T)}{f_2(T)}\frac{d^2T}{dx^2}+\frac{\chi}{\dot{f}_1(R)f_2(T)}[T_3^3-T_1^1]$$

$$+\frac{f_1(R)\dot{f}_2(T)}{\dot{f}_1(R)f_2(T)}[(T_1^1+\theta_1^1)-(T_3^3+\theta_3^3)] \quad (5.11c)$$

By using (5.9) and (3.8), the equations (5.11) reduces to

$$\frac{\ddot{B}}{B}-\frac{\ddot{A}}{A}+\frac{\dot{C}}{C}\left[\frac{\dot{B}}{B}-\frac{\dot{A}}{A}\right]+\left(\frac{\dot{B}}{B}-\frac{\dot{A}}{A}\right)\left[\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}+\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dt}\right]=\frac{1}{A^2}\left[\frac{\ddot{f}_2(T)}{f_2(T)}\left(\frac{dT}{dx}\right)^2+\frac{\dot{f}_2(T)}{f_2(T)}\frac{d^2T}{dx^2}-m\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dx}\right] \quad (5.12a)$$

$$\frac{\ddot{C}}{C}-\frac{\ddot{B}}{B}+\frac{\dot{A}}{A}\left[\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\right]+\left(\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\right)\left[\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}+\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dt}\right]=-\frac{m^2}{A^2}+\frac{m}{A^2}\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dx} \tag{5.12b}$$

$$\frac{\ddot{A}}{A}-\frac{\ddot{C}}{C}+\frac{\dot{B}}{B}\left[\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right]+\left(\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right)\left[\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}+\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dt}\right]=\frac{m^2}{A^2}-\frac{1}{A^2}\frac{\ddot{f}_2(T)}{f_2(T)}\left(\frac{dT}{dx}\right)^2-\frac{1}{A^2}\frac{\dot{f}_2(T)}{f_2(T)}\frac{d^2T}{dx^2} \tag{5.12c}$$

With the help of (5.5) we can write the equations (3.9) as

$$\left(\frac{\dot{h}_7}{h_7}\right)^{\cdot}+\frac{\dot{h}_7^{\,2}}{h_7^{\,2}}+\frac{\dot{h}_7}{h_7}\left[\frac{\dot{C}}{C}+\frac{\dot{B}}{B}-\frac{\dot{A}}{A}\right]=\dot{\psi}\dot{\varphi}^2 \tag{5.13a}$$

$$\left(\frac{\dot{h}_7}{h_7}\right)^{\cdot}+\frac{\dot{h}_7^{\,2}}{h_7^{\,2}}+\frac{\dot{h}_7}{h_7}\left[\frac{\dot{A}}{A}+\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\right]=\dot{\psi}\dot{\varphi}^2 \tag{5.13b}$$

$$\left(\frac{\dot{h}_7}{h_7}\right)^{\cdot}+\frac{\dot{h}_7^{\,2}}{h_7^{\,2}}+\frac{\dot{h}_7}{h_7}\left[\frac{\dot{B}}{B}+\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right]=\dot{\psi}\dot{\varphi}^2 \tag{5.13c}$$

These equations further imply that

$$\frac{\dot{C}}{C}+\frac{\dot{B}}{B}-\frac{\dot{A}}{A}=\frac{\dot{A}}{A}+\frac{\dot{C}}{C}-\frac{\dot{B}}{B}=\frac{\dot{B}}{B}+\frac{\dot{A}}{A}-\frac{\dot{C}}{C}$$

or $$\frac{\dot{A}}{A}=\frac{\dot{B}}{B}=\frac{\dot{C}}{C} \tag{5.14}$$

Upon integration the equation (5.14) yields

$$A=m_9B,\; B=m_{10}\,C\,,\;\; C=m_{11}A \tag{5.15}$$

where m′s are constants of integration.

We observe that *A* is scalar multiple of *B*, *B* is scalar multiple of C and *C* is scalar multiple of *A*

By using (514) the R. H. S. of (5.12) vanishes for all t

Therefore for solving differential equation of *A*, *B*, *C* with respect to t, we consider the L.H.S. of equations (5.12)

$$\frac{\ddot{B}}{B}-\frac{\ddot{A}}{A}+\frac{\dot{C}}{C}\left[\frac{\dot{B}}{B}-\frac{\dot{A}}{A}\right]+\left(\frac{\dot{B}}{B}-\frac{\dot{A}}{A}\right)\left[\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}+\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dt}\right]=0 \tag{5.16a}$$

$$\frac{\ddot{C}}{C}-\frac{\ddot{B}}{B}+\frac{\dot{A}}{A}\left[\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\right]+\left(\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\right)\left[\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}+\frac{\dot{f}_2(T)}{f_2(T)}\frac{d.}{dt}\right]=0 \tag{5.16b}$$

$$\frac{\ddot{A}}{A}-\frac{\ddot{C}}{C}+\frac{\dot{B}}{B}\left[\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right]+\left(\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right)\left[\frac{\ddot{f}_1(R)}{\dot{f}_1(R)}\frac{dR}{dt}+\frac{\dot{f}_2(T)}{f_2(T)}\frac{dT}{dt}\right]=0 \tag{5.16c}$$

Integrating (5.16) we obtain

$$\frac{B}{A} = m_{13}\exp\left\{m_{12}\int \frac{1}{ABC\dot{f}_1(R)f_2(T)}dt\right\} \tag{5.17a}$$

$$\frac{A}{C} = m_{15}\exp\left\{m_{14}\int \frac{1}{ABC\dot{f}_1(R)f_2(T)}dt\right\} \tag{5.17b}$$

$$\frac{C}{B} = m_{17}\exp\left\{m_{16}\int \frac{1}{ABC\dot{f}_1(R)f_2(T)}dt\right\} \tag{5.17c}$$

where m's are constants of integration, such that

$$m_{13}m_{15}m_{17} = 1 \text{ and } m_{12} + m_{14} + m_{16} = 0$$

From (5.17) we can express explicitly the values of A, B, C as

$$A = (ABC)^{\frac{1}{3}}m_{18}\exp\left\{m_{19}\int \frac{1}{ABC\dot{f}_1(R)f_2(T)}dt\right\} \tag{5.18a}$$

$$C = (ABC)^{\frac{1}{3}}m_{20}\exp\left\{m_{21}\int \frac{1}{ABC\dot{f}_1(R)f_2(T)}dt\right\}$$
(5.18b)

$$B = (ABC)^{\frac{1}{3}}m_{22}\exp\left\{m_{23}\int \frac{1}{ABC\dot{f}_1(R)f_2(T)}dt\right\} \tag{5.18c}$$

where m's are constants of integration.

Adjusting the constants in (4.17) and (5.18), the line element (3.1) assumes an isotropic form and hence we generalize the result in the form of following theorem.

Theorem 1: In $f(R, T)$ theory of gravity, the Bianchi type III space-time filled with scalar field coupled with electromagnetic field, admits isotropy for the functional form
$f(R,T) = f_1(R) + \lambda f_2(T)$ *and* $f(R,T) = f_1(R)f_2(T)$.

Inserting (5.14) in (5.13), we get

$$\left(\frac{\dot{h}_7}{h_7}\right)^{\cdot} + \frac{\dot{h}_7{}^2}{h_7{}^2} + \frac{\dot{h}_7}{h_7}\left[\frac{\dot{A}}{A}\right] = \dot{\psi}\dot{\varphi}^2 \tag{5.19}$$

But from (5.6) we have

$$\dot{\psi}\dot{\varphi}^2 = \frac{\dot{h}_7{}^2}{h_7{}^2} + \frac{f_1(R)\dot{f}_2(T)}{\chi}I\ddot{\psi}\dot{\varphi}^2 \tag{5.20}$$

Inserting (5.20) in (5.19), we get

$$\left(\frac{\dot{h}_7}{h_7}\right)^{\cdot} + \frac{\dot{h}_7}{h_7}\left[\frac{\dot{A}}{A}\right] = \frac{\lambda}{\chi}I\ddot{\psi}\dot{\varphi}^2 \tag{5.21}$$

Confining to the linearity of the function ψ i.e.$\ddot{\psi} = 0$ or $\psi = m_{24}I + m_{25}$ then (5.21) has solution

$$h_7 = m_{27} exp\left\{m_{26} \int \frac{1}{A} dt\right\} \tag{5.22}$$

With the help of (5.22), the equations (5.7) convert in to

$$V_1 = m_{28} exp\left\{m_{26} \int \frac{1}{A} dt\right\} \tag{5.23a}$$

$$V_2 = m_{29} exp\left\{m_{26} \int \frac{1}{A} dt\right\} \tag{5.23b}$$

$$V_3 = m_{30} exp\left\{m_{26} \int \frac{1}{A} dt\right\} \tag{5.23c}$$

where m′s are constants of integration.

Adjusting the constants in (4.22) and (5.23) the vector potential assume the following form

$$V_i = [V_1, V_1, v_1, 0]$$

Hence we generalize the result in the form of the following theorem.

Theorem 2: : In f(R, T) theory of gravity, the Bianchi type III space-time filled with scalar field coupled with electromagnetic field, admits the vector potential $V_i = [V_1, V_1, v_1, 0]$ for the functional form $f(R,T) = f_1(R) + \lambda f_2(T)$ and $f(R,T) = f_1(R) f_2(T)$**.**

6. Sub case $f(R,T) = f(R)$

In this case we follow the notations

$$f(R,T) = f(R), \quad f_R(R,T) = \frac{\partial f(R,T)}{\partial R} = \dot{f}(R), \quad f_T(R,T) = \frac{\partial f(R,T)}{\partial T} = 0$$

In this case the field equations (1.8) reduces to

$$G_j^i = \frac{1}{\dot{f}(R)}\left[g^{im}\nabla_m\nabla_j \dot{f}(R)\right] - \frac{1}{6\dot{f}(R)}\left[\dot{f}(R)R + f(R)\right]g_j^i + \frac{\chi}{\dot{f}(R)}\left[T_j^i - \frac{1}{3}T g_j^i\right] \tag{6.1}$$

The computation for this case easily follows from those of the earlier case (section 4) by mere substitution of $f_1(R) = f(R), \lambda = 0$ or $f_2(T) = 0$

We get the result as follows

$$A = (ABC)^{\frac{1}{3}} k_{45} exp\left\{k_{41} \int \frac{1}{ABC\dot{f}_1(R)} dt\right\} \tag{6.2a}$$

$$B = (ABC)^{\frac{1}{3}} k_{46} exp\left\{k_{43} \int \frac{1}{ABC\dot{f}_1(R)} dt\right\} \tag{6.2b}$$

$$C = (ABC)^{\frac{1}{3}} k_{47} exp\left\{k_{41} \int \frac{1}{ABC\dot{f}_1(R)} dt\right\} \tag{6.2c}$$

where k′s are constant of integration.

$$\mathrm{V_1 = k_{50} exp\left\{k_{48} \int \frac{1}{A} dt\right\}} \tag{6.3a}$$

$$\mathrm{V_2 = k_{51} exp\left\{k_{48} \int \frac{1}{A} dt\right\}} \tag{6.3b}$$

$$\mathrm{V_3 = k_{52} exp\left\{k_{48} \int \frac{1}{A} dt\right\}} \tag{6.3c}$$

where k′s are constant of integration.

From section 4, 5 and 6 we observe that the result remain intact for $f(R,T) = f_1(R) + \lambda f_2(T)$, $f(R,T) = f_1(R) f_2(T)$ and $f(R,T) = f(R)$ differ in constant of integration only. Hence the equations (6.2) and (6.3) admit the theorem 1 and 2.

7. Sub case $\boldsymbol{f(R,T) = R + \lambda T}$

In this case we follow the notations

$$f_R(R,T) = \frac{\partial f(R,T)}{\partial R} = 1 \quad , \qquad f_T(R,T) = \frac{\partial f(R,T)}{\partial T} = \lambda$$

The field equation (1.5) reduces to

$$G_j^i = \chi T_j^i - \lambda\left[T_j^i + \theta_j^i\right] + \frac{\lambda}{2} T \delta_j^i \tag{7.1}$$

The computation of this case follows from section 4, $f(R,T) = f_1(R) + \lambda f_2(T)$ by taking $f_1(R) = R$ and $f_2(T) = T$

We get the result as follows

$$\mathrm{A = (ABC)^{\frac{1}{3}} l_{17} exp\left\{l_{13} \int \frac{1}{ABC} dt\right\}} \tag{7.2a}$$

$$B = (ABC)^{\frac{1}{3}} l_{18} \exp\left\{l_{15} \int \frac{1}{ABC} dt\right\}$$
(7.2b)

$$C = (ABC)^{\frac{1}{3}} l_{19} \exp\left\{l_{13} \int \frac{1}{ABC} dt\right\} \quad (7.2c)$$

wherel′sare constants of integration.

$$V_1 = l_{24} \exp\left\{l_{22} \int \frac{1}{A} dt\right\} \quad (7.3a)$$

$$V_2 = l_{25} \exp\left\{l_{22} \int \frac{1}{A} dt\right\} \quad (7.3b)$$

$$V_3 = l_{26} \exp\left\{l_{22} \int \frac{1}{A} dt\right\} \quad (7.3c)$$

where l′s are constant of integration.

From section 4, 5 and 7 we observe that the result remain intact for $f(R,T) = f_1(R) + \lambda f_2(T)$, $f(R,T) = f_1(R) f_2(T)$ and $f(R,T) = R + \lambda T$, differ in constant of integration only. Hence the equations (7.2) and (7.3) admit the theorem 1 and 2.

8. Sub case $f(R,T) = f(R) = R$

In this case $f(R,T) = f(R) = R$, $f_R(R,T) = 1$, $f_T(R,T) = 0$

Then field equation (1.5) reduces to

$$G^i_j = \chi T^i_j \quad (8.1)$$

The computation for this case easily follows from those of the earlier case section 7 by mere substitution of $\lambda = 0$ we get the result as follows

$$A = (ABC)^{\frac{1}{3}} l_{43} \exp\left\{l_{39} \int \frac{1}{ABC} dt\right\} \quad (8.2a)$$

$$B = (ABC)^{\frac{1}{3}} l_{44} \exp\left\{l_{41} \int \frac{1}{ABC} dt\right\} \quad (8.2b)$$

$$C = (ABC)^{\frac{1}{3}} l_{45} \exp\left\{l_{39} \int \frac{1}{ABC} dt\right\} \quad (8.2c)$$

where l′s are constants of integration.

$$V_1 = l_{48} \exp\left\{l_{46} \int \frac{1}{A} dt\right\} \quad (8.3a)$$

$$V_2 = l_{49} \exp\left\{l_{46} \int \frac{1}{A} dt\right\} \tag{8.3b}$$

$$V_3 = l_{50} \exp\left\{l_{46} \int \frac{1}{A} dt\right\} \tag{8.3c}$$

where l's are constant of integration.

From section 4, 5 and 8 we observe that the result remain intact for $f(R,T) = f_1(R) + \lambda f_2(T)$, $f(R,T) = f_1(R) f_2(T)$ and $f(R,T) = R$, differ in constant of integration only. Hence the equations (8.2) and (8.3) admit the theorem 1 and 2.

9. Consideration of particular case $f(R,T) = RT$

In this case $f_R(R,T) = T$, $f_T(R,T) = R$

Then the field equation (1.8) reduces to

$$G_j^i = \frac{1}{T}\left[g^{im} \nabla_m \nabla_j T\right] - \frac{R}{3} g_j^i + \frac{\chi}{T}\left[T_j^i - \frac{1}{3} T g_j^i\right] + \frac{1}{3}\frac{R}{T}[T+\theta] g_j^i - \frac{R}{T}\left[T_j^i + \theta_j^i\right] \tag{9.1}$$

The computation for this case easily follows from those of the earlier case, section 5, by mere substitution of $f_1(R) = R$ and $f_2(T) = T$

We get the result as follows

$$A = (ABC)^{\frac{1}{3}} n_{13} \exp\left\{n_{14} \int \frac{1}{ABCT} dt\right\} \tag{9.2a}$$

$$B = (ABC)^{\frac{1}{3}} n_{15} \exp\left\{n_{16} \int \frac{1}{ABCT} dt\right\} \tag{9.2b}$$

$$C = (ABC)^{\frac{1}{3}} n_{17} \exp\left\{n_{18} \int \frac{1}{ABC\dot{f}_1(R) f_2(T)} dt\right\} \tag{9.2c}$$

where n's are constants of integration.

$$V_1 = n_{23} \exp\left\{n_{21} \int \frac{1}{A} dt\right\} \tag{9.3a}$$

$$V_2 = n_{24} \exp\left\{n_{21} \int \frac{1}{A} dt\right\} \tag{9.3b}$$

$$V_3 = n_{25} \exp\left\{n_{21} \int \frac{1}{A} dt\right\} \tag{79.3c}$$

where n's are constants of integration.

From section 4, 5 and 9 we observe that the result remain intact for $f(R,T) = f_1(R) + \lambda f_2(T)$, $f(R,T) = f_1(R)f_2(T)$ and $f(R,T) = RT$, differ in constant of integration only. Hence the equations (9.2) and (9.3) admit the theorem 1 and 2.

10. Conclusion

i) In the present paper we have considered sub cases of $\mathrm{f(R,T)}$ theory of gravity models $\mathrm{f(R,T) = f_1(R) + \lambda f_2(T)}$, $\mathrm{f(R,T) = f(R)}$, $\mathrm{f(R,T) = R + \lambda T}$, $\mathrm{f(R,T) = f_1(R)f_2(T)}$, $\mathrm{f(R,T) = RT}$ in Bianchi type III metric. We have derived the gravitational field equations corresponding to the general and particular cases of $\mathrm{f(R,T)}$ theory of gravity.

ii) It is observed that, even though the cases of $\mathrm{f(R,T)}$ theory are distinct, the convergent, non-singular, isotropic solutions can be evolved in each case along with the components vector potential.

iii) From finding of the $\mathrm{f(R,T)}$ and$\mathrm{f(R)}$ theory, general and particular cases, in this paper we believe firmly that the results of $\mathrm{f(R,T)}$ and $\mathrm{f(R)}$ depends on only R and not on T.

iv) From different cases of $\mathrm{f(R,T)}$ we observe that the results remain intact only differ in constants of integration.

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References

[1] BijanSaha, Int. Theor. Phys. DOI 10.1007/s 10773-015-2615-1, 31 March 2015
[2] Clocchiatti A et al. (High Z S N Search Collaboration); Astrophys. J., **642**,1,2006
[3] Harko T, Lobo F S N, Nojiri S, Odintsov S D: Phys. Rev. D **84**,024020 (2011)
[4] MyrzakulovRatbay,ErtanGudekli, Razina Olga and TsybaPyotr : Int. J. Of Maths.Computational, Natural and Physical Engineering, **9,** 2, 2015
[5] Nojiri S and Odintsov S D : Phys. Rev. D 68, 123512 (2003).
[6] Perlmeter S et al. (Supernova Cosmological Project Collaboration); Nature **391**, 51, 1998
[7] Perlmutter S et al. Astrophys. J., **598**,102, 2003
[8] Rybakov Yu. P, Shikin G N and SahaB : Int. J. Theor. Phys**. 36**, 6,pp. 1475-1494, 1997
[9] Riess A G et al. (Supernova Search Team Collaboration); Astron. J., **116**, 1009, 1998
[10] Riess A G et al. (Supernova Search Team Collaboration); Astrophys. J., 607, **665**, 2004
[11] Tonry J L et al. (Supernova Search Team Collaboration); Astrophys. J., **594**,1,2003