Article

On Some Expressions for ζ(*k*), *k* = 2, 3, 4

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Abstract

We obtain the hypergeometric form of $\sum_{r=1}^{\infty} \frac{1}{r^m \binom{2r}{r}}$, $m = 2, 3, 4$.

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1. Introduction

In the literature [1-6] we find the following expressions for three values of the Riemann zeta function [7, 8]:

$$S_2 \equiv \sum_{r=1}^{\infty} \frac{1}{r^2 \binom{2r}{r}} = \frac{1}{3}\zeta(2), \quad S_3 \equiv \sum_{r=1}^{\infty} \frac{1}{r^3 \binom{2r}{r}} = \frac{2}{5}\zeta(3), \quad S_4 \equiv \sum_{r=1}^{\infty} \frac{1}{r^4 \binom{2r}{r}} = \frac{17}{36}\zeta(4), \tag{1}$$

here we shall show their hypergeometric version. It is interesting to note that this formula for ζ(3) is important to prove its irrationality [9-11]; we know that [12-15] $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$, hence $S_2 = \frac{\pi^2}{18}$ and $S_4 = \frac{17\,\pi^4}{3240}$.

2. Formulae of Euler, Hjortnaes, Melzak, Comtet and Apéry

From (1):

$$S_2 = \frac{1}{2}\sum_{k=0}^{\infty} t_k, \qquad t_k = \frac{2}{(k+1)^2 \binom{2k+2}{k+1}}, \qquad t_0 = 1, \qquad \frac{t_{k+1}}{t_k} = \frac{(k+1)^2}{4(k+2)\,(k+\frac{3}{2})},$$

then it is immediate [16-18] the hypergeometric form:

$$S_2 = \frac{1}{2}\ {}_3F_2(1, 1, 1;\ 2,\ \tfrac{3}{2};\ \tfrac{1}{4})\,. \tag{2}$$

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Similarly, for $m = 3, 4$:

$$S_m = \frac{1}{2}\sum_{k=0}^{\infty} t_k \ , \qquad t_k = \frac{2\,(-1)^{mk}}{(k+1)^m \binom{2k+2}{k+1}} \ , \qquad t_0 = 1, \qquad \frac{t_{k+1}}{t_k} = \frac{(-1)^m\,(k+1)^m}{4(k+2)^{m-1}\,(k+\frac{3}{2})} \ , \tag{3}$$

therefore:

$$S_3 = \frac{1}{2}\ {}_4F_3\left(1,1,1,1;\ 2,2,\ \frac{3}{2};\ -\frac{1}{4}\right), \qquad S_4 = \frac{1}{2}\ {}_5F_4\left(1,1,1,1,1;2,2,2,\frac{3}{2};\ \frac{1}{4}\right), \tag{4}$$

and Mathematica gives the values $\frac{\pi^2}{9}$, $\frac{4}{5}\zeta(3)$ and $1.022194165\ldots = \frac{17\,\pi^4}{1620}$ for the hypergeometric functions ${}_3F_2$, ${}_4F_3$ and ${}_5F_4$, respectively, present at (2) and (4), which is equivalent to (1).

In [3] the authors indicate that Euler [19] deduced the expression:

$$\sum_{r=1}^{\infty} \frac{(2x)^{\,2r}}{r^2\binom{2r}{r}} = 2\,(arc\sin x)^2 \ , \tag{5}$$

that for $x = \frac{1}{2}$ implies the relation (1) for S_2 [1, 20]; van der Poorten [11] comments that the formula (1) for S_3 obtained by Apéry [1] was proved by Hjortnaes [21]. Our procedure can be applied to show diverse identities, for example [1, 3, 5, 6]:

$$\sum_{r=1}^{\infty} \frac{1}{\binom{2r}{r}} = \frac{1}{3} + \frac{2\sqrt{3}\,\pi}{27} \ , \ \sum_{r=1}^{\infty} \frac{1}{r\binom{2r}{r}} = \frac{\sqrt{3}\,\pi}{9} \ , \ \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r\binom{2r}{r}} = \frac{2\,Ln\,\phi}{\sqrt{5}} \ , \ \sum_{r=1}^{\infty} \frac{3^r}{r^2\binom{2r}{r}} = \frac{2\,\pi^2}{9}, \text{etc.} \tag{6}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio [22].

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