Article

Hypergeometric Form of the k-Fibonacci Numbers

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Abstract

The *k*-Fibonacci sequence is written in terms of the Gauss hypergeometric function.

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1. Introduction

The Fibonacci numbers F_n are defined via the recurrence relation [1]:

$$F_{n+1} = F_n + F_{n-1}\,, \qquad F_0 = 0, \qquad F_1 = 1, \qquad n \geq 1, \tag{1}$$

thus $\{F_n\} = \{0, 1, 1, 2, 3, 5, 8, \dots\}$. Now we introduce the *k*-Fibonacci numbers [2]:

$$F_{k,n+1} = k\, F_{k,n} + F_{k,n-1}\,, \qquad F_{k,0} = 0, \qquad F_{k,1} = 1, \qquad n, k = 1, 2, 3, \dots \tag{2}$$

hence $F_{1,n} = F_n$, and the solution of (2) gives the following two equivalent expressions for $n \geq 2$ [2]:

$$F_{k,n} = \frac{1}{2^{n-1}} \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2m+1} k^{n-1-2m} (k^2+4)^m\,, \tag{3}$$

$$F_{k,n} = \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-m}{m} k^{n-1-2m}\,, \tag{4}$$

where $\lfloor A \rfloor$ is the floor function of A.

In the next Section we employ the technique of [3-5] to obtain the hypergeometric version of (3) and (4).

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2. *k*-Fibonacci sequence in terms of the Gauss hypergeometric function

The relation (3) can be written in the form:

$$F_{k,n} = \frac{n\,k^{n-1}}{2^{n-1}} \sum_{m=0}^{\infty} t_m\,, \qquad t_m = \frac{1}{n\,k^{n-1}} \binom{n}{2m+1} k^{n-1-2m}\,(k^2+4)^m\,, \tag{5}$$

therefore $t_0 = 1$ and:

$$\frac{t_{m+1}}{t_m} = \frac{\left(m+\frac{1-n}{2}\right)\left(m+\frac{2-n}{2}\right) k^{n-2}\,(k^2+4)}{\left(m+\frac{3}{2}\right)(m+1)}\,,$$

then from [3-5] it is immediate that (5) acquires the following structure in terms of the Gauss hypergeometric function:

$$F_{k,n} = \frac{n\,k^{n-1}}{2^{n-1}}\; {}_2F_1\left(\frac{1-n}{2},\ \frac{2-n}{2};\ \frac{3}{2};\ (k^2+4)\,k^{n-2}\right); \tag{6}$$

thus, for $k = 1$ we deduce the formula of Dilcher [6, 7] for the Fibonacci numbers:

$$F_n = F_{1,n} = \frac{n}{2^{n-1}}\; {}_2F_1\left(\frac{1-n}{2}\,,\ \frac{2-n}{2};\ \frac{3}{2};5\right). \tag{7}$$

Similarly, the expression (4) adopts the form:

$$F_{k,n} = k^{n-1} \sum_{m=0}^{\infty} t_m\,, \qquad t_m = \frac{1}{k^{2m}}\binom{n-1-m}{m}, \qquad t_0 = 1, \tag{8}$$

such that:

$$\frac{t_{m+1}}{t_m} = \frac{\left(m+\frac{1-n}{2}\right)\left(m+\frac{2-n}{2}\right)(-4)}{(m+1-n)\,(m+1)\,k^2}\,,$$

hence (8) takes the hypergeometric structure:

$$F_{k,n} = k^{n-1}\; {}_2F_1\left(\frac{1-n}{2}\,,\ \frac{2-n}{2}\,;\ 1-n\,;\ -\frac{4}{k^2}\right), \tag{9}$$

thus $F_n = {}_2F_1\left(\frac{1-n}{2}\,,\frac{2-n}{2}\,;\ 1-n\,;\ -4\right)$.

With (6) or (9) is simple to calculate the Pell sequence $\{F_{2,n}\} = \{0, 1, 2, 5, 12, 29, 70, \ldots\}$, and $\{F_{3,n}\} = \{0, 1, 3, 10, 33, 109, \ldots\}$ in according with the results of [1].

Remark.- The identity [4]:

$$\frac{1}{2^{n-1}} \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2m+1} (1+4x)^m = \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-m}{m} x^m , \qquad (10)$$

with $x = k^{-2}$ allows to show that (3) is equivalent to (4).

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